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A PRACTICAL GPS NETWORK ADJUSTMENT METHOD

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ABSTRACT

The partial derivatives of space coordinates to geodetic coordinates are numerically constructed in a GPS network adjustment. The plane coordinates and ellipsoidal heights are selected as parameters to be adjusted. These parameters can easily be modified to accommodate plane or height networks or to accept additional observations. The model is demonstrated using real GPS data and verified by comparing the results against a solution obtained from a rigorous model.

INTRODUCTION

GPS has become the most preferred method to establish a control network. The general data processing approach is first to calculate the baseline vectors using the receiver manufacturer's software. The plane coordinates and the ellipsoidal heights of known points are then converted to space coordinates. Finally, adjustment is carried out in the space coordinate system from which plane coordinates of adjusted stations are obtained referring to a datum using map projection and datum parameters. Alternatively, the adjustment can be carried out in a two-dimensional plane coordinate system after the baseline vectors have been projected using the adopted map projection.

In this paper, a practical network adjustment method is introduced. The model assumes baseline vectors as observations and selects plane coordinates and ellipsoidal heights of stations as parameters to be estimated. The formulation is such that additional observations, such as eccentric data or classical plane surveying observations, can easily be introduced. In addition, if the height parameters are eliminated from the normal equation, network adjustment transforms to the adjustment of a plane network.

In this model, partial derivatives of space coordinates with respect to plane coordinates and heights are needed. This formulation is rarely used because of the complexity of constructing the partial derivatives in linearizing the observation equations. This study introduces a numerical approach to facilitate the solution of this problem.

ADJUSTMENT MODEL AND DISCUSSION

Adjustment Model

Let subscripts i and k denote the station numbers, $\mathbf{R} = (X, Y, Z)^T$ represents the space position vector of the station, and $\Delta\mathbf{R}$ the baseline vector which is the output of the GPS baseline processing software. Then the observation equation of baseline j from station i to station k can be expressed as,

$$V_j + \Delta R_j = R_i - R_k \tag{1}$$

where $\mathbf{V}_j$ is the baseline vector residual with a weight matrix $\mathbf{P}_j$ which is obtained from baseline processing software together with the baseline vectors.

If $\mathbf{R}^0$ denotes the approximate value of a station position in space coordinates, then $\mathbf{R}$ in equation (1) can be rewritten as

$$R = R^0 + \delta R \tag{2}$$

In this expression, the correction vector $\delta\mathbf{R}$ to the space position vector can also be represented by the correction to the corresponding plane position coordinates and the ellipsoidal height, $\delta\mathbf{R}'$. i.e.,

$$\delta R = \frac{\partial R}{\partial r'} \delta r' \tag{3}$$

where $\mathbf{r}' = (x'\ y'\ H')$ is the vector which consists of the plane coordinates and the ellipsoidal height of the station, and $\partial\mathbf{R}/\partial\mathbf{r}'$ (Jacobean) is the partial derivative of the space coordinates of the station with respect to the plane coordinates and the ellipsoidal height. The calculation of the Jacobean will be examined in the following section.

The results of a control network solution need to be expressed in a local plane coordinate system. In general, a four parameter similarity transformation (one scale, two shift, and one rotation) is needed for transformation between the local coordinate system and the plane coordinates that are obtained by projecting GPS originated space coordinates using a map projection. The transformation between two plane coordinate systems, referring to a rotation about a reference point, can be expressed as

$$\begin{pmatrix} x'-x_0' \\ y'-y_0' \end{pmatrix} = (1+k)\theta(\alpha)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \tag{4}$$

where, k and α are the scale factor and the rotation angle respectively, (x, y) is the position vector of a station in the local plane coordinate system, and (x_0, y_0) is the position vector of a rotation reference station (or an average location) whose value will be maintained the same after rotation as before rotation. The rotation matrix in equation (4) is given by

$$\theta(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{5}$$

Note that the shift parameters do not appear on the right-hand side of equation (4) because only coordinate differences are of interest.

Considering equations (1) to (5), we have

$$\boldsymbol{V}_j = \boldsymbol{D}_i \delta \boldsymbol{r}_i - \boldsymbol{D}_k \delta \boldsymbol{r}_k + (\boldsymbol{E}_i - \boldsymbol{E}_k)\begin{pmatrix} \delta k \\ \delta \alpha \end{pmatrix} - \boldsymbol{l}_j \tag{6}$$

where

$$\boldsymbol{D}_i = \frac{\partial \boldsymbol{R}_i}{\partial \boldsymbol{r}_i'} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{E}_i = \frac{\partial \boldsymbol{R}_i}{\partial \boldsymbol{r}_i'} \left(\theta(\alpha) \begin{pmatrix} x-x_0 \\ y-y_0 \\ 0 \end{pmatrix} \frac{d\theta(\alpha)}{d\alpha} \begin{pmatrix} x-x_0 \\ y-y_0 \\ 0 \end{pmatrix} \right)$$

$$\frac{d\theta(\alpha)}{d\alpha} = \begin{pmatrix} -\sin(\alpha) & -\cos(\alpha) \\ \cos\alpha & -\sin(\alpha) \end{pmatrix}$$

$$l_j = \Delta R_j + R_i^0 - R_k^0 - (E_i - E_k)\begin{pmatrix} k^0 \\ \alpha^0 \end{pmatrix}$$

In the above equations, k^0 and α^0 are the approximate values for the scale factor and rotation angle. Observing that the weight matrix of equation (6) is $\mathbf{P}_j$, the normal equations, using the linearized observation equations given in equation (6), can be formed for each observed baseline vector as,

$$\mathbf{N}\ \delta\mathbf{x} = \mathbf{C} \tag{7}$$

In equation (7), **N** and **C** are the normal matrix and the constant term of the normal equation, $\delta\mathbf{x}$ is the correction to the parameters to be estimated. If n is the number of stations then the dimension of the correction vector to the parameters to be estimated is $3 \times n + 2$. ($3n$ for the station positions and 2 for the scale and the rotation parameters).

Fixing the coordinates and heights of some stations to their *a priori* known values can lead to the solution of the normal equation. In order to solve the normal equations using station coordinates together with the scale factor and rotation angle, at least plane coordinates of two stations and the height of one station should be known. If the scale factor and rotation angle are not to be solved, the plane coordinates and the height of only one point are needed for the solution.

Partial derivatives of space coordinates to plane coordinates and height.

As mentioned above, the analytical expression for the Jacobean $\partial\mathbf{R}/\partial\mathbf{r}'$ is relatively complicated because the calculation of geodetic coordinates (latitude, longitude and height) from space coordinates is an iterative procedure (exact solutions are also too complicated) and the calculation of plane coordinates from latitude and longitude involves a map projection with a multitude of terms. Here, a simple numerical method will be introduced to calculate the partial derivatives.

At the station i, let $\partial\mathbf{R}/\partial\mathbf{r}'$ be expressed as,

$$\left(\frac{\partial R}{\partial r'}\right)_i = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_i \tag{8}$$

which means that

$$\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix}_i = \left(\frac{\partial R}{\partial r'}\right)_i \begin{pmatrix}\delta x'\\ \delta y'\\ \delta H\end{pmatrix}_i = \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}_i \begin{pmatrix}\delta x'\\ \delta y'\\ \delta H\end{pmatrix}_i$$

Making use of the above relationships, the corresponding space coordinates $\mathbf{R}_i^0 = (X_i^o\ Y_i^o\ Z_i^o)$ can be calculated from the approximate values of plane coordinates and ellipsoidal height of station i, $\mathbf{r}_i'^o = (x_i'^o\ y_i'^o\ H_i^o)$. First, by adding a small change d to $x_i'^o$, the changes of space coordinates $(\Delta X\ \Delta Y\ \Delta Z)$ are calculated. Then, from equation (8), the following coefficients are determined,

$$a_{11} = \frac{\Delta X}{d}\ ,\quad a_{21} = \frac{\Delta Y}{d}\ ,\quad a_{31} = \frac{\Delta Z}{d} \tag{9}$$

Similarly, other components of $\partial\mathbf{R}/\partial\mathbf{r}'$ are obtained, by adding small changes to $y_i'^o$ and $z_i'^o$.

By definition, should d go to zero in the limit, the above approach is rigorous. Nevertheless, because the observation equations are non-linear, an iterative solution to the problem is needed. Therefore, only the first two digits for each component of the partial derivatives are sufficient in forming the normal equations. Numerical tests show that 0.1m for d is sufficient for computing the partial derivatives without causing any numerical instabilities.

Discussion of the adjustment model

The above model is suitable for most GPS control networks. Nevertheless, in some cases, if the rotation angle is large, the estimate for the scale factor k may be biased by the rotation angle or vice versa because of the high correlation between these two variables. A numerical solution showed that when rotation angle is about 1°, k changes 150ppm. The solution to this problem is first to rotate the known points to the GPS defined direction, then to rotate the adjusted results of each station back. In other words, all observed baseline vectors are adjusted in the space coordinate system by fixing one GPS station to its single point positioning result. The adjusted space coordinates of the stations, which are known in the local coordinate system, are projected to the map projection plane according to the adopted map projection. By comparing the projected plane coordinates and known coordinates, the rotation angle and shift are determined. After rotating and shifting the coordinates of known points in the local coordinate system with the rotation angle and shift, network adjustment is carried out with the above model and final results are obtained by rotating and shifting the plane coordinates back to the local plane coordinate system.

Because the parameters in adjustment are plane coordinates and ellipsoidal control network.

Let $\xi(x\ y\ \mathbf{p})$ be the geoid undulation model for the local area (a surface polynomial for instance). Then the geometric height (ellipsoidal height) at station i can be expressed as

$$H_i = h_i + \xi(x_i\ y_i\ \mathbf{p}) \tag{10}$$

where h_i is the levelled (orthometric) height, and p is the parameter in the geoid undulation model to be estimated.

From equation (10) we have

$$\delta H_i = \delta h_i + \frac{\partial \xi}{\partial p}\delta \mathbf{p} \tag{11}$$

By applying equation (11) to equation (6), we can transform the adjustment problem to include plane coordinates, levelled height, scale factor, rotation angle and geoid undulation as model parameters to be estimated. It should be pointed out that if the geoid undulations are not represented well by the model, they will contaminate the other parameters.

Example and Conclusion

One GPS control network was composed of 119 points and the receivers used were Ashtech, Leica, Trimble, Sokkia, etc. Using a commercial GPS network adjustment software, TGPPS **[1]**, which is widely used in China, the rotation angle was estimated to be about 1° as shown in Table 1. When compared to the model solution results presented in this study, the estimated plane coordinates, ellipsoidal heights, rotation angle, and their standard errors were practically the same. The scale factor is changed from 164ppm, which is distorted by a big rotation angle, to about 19ppm.

Table 1. *Comparison of scale factor and rotation angle results obtained from TGPS and the proposed model.*

	With TGPPS	With introduced model
Rotation angle	0° 58′ 36.73″	0° 58′ 37.37″
Scale factor	-164.29	-18.92

These results show that the model is convenient as a practical GPS network adjustment procedure. The model easily accepts other observations such as eccentric data and forming the normal equations via partial derivatives greatly simplifies the model.

References

1. Jin Guoxiong, et al. 1994. *Applications and Data Processing of GPS Positioning*. Tongji Publishing Press, Peoples Republic of China.