A Reduced-Order Model

for

Integrated GPS/INS

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ABSTRACT

The dominant factor in determining the computation time of the Kalman filter is the dimension n of the model state vector. The number of computations per iteration is on the order of n^3. Any reduction in the number of states will benefit directly in terms of increased computation time. In this paper, a high order model in integrated GPS/INS is described first, then a reduced-order model based on the high-order model, is developed. Finally, a faster tracking approach for Kalman filters is discussed.

A typical aircraft trajectory is designed for a complex high-dynamic aircraft flight experiment. A Monte Carlo analysis shows that the reduced order model presented in this paper provides satisfactory accuracy for aircraft navigation.

INTRODUCTION

Kalman filters have been widely used in the integration of the GPS/INS system, both the dynamic model and observation model are required to describe the integrated system. The dynamic model is usually described by a linear differential equation involving the system errors of the INS and GPS systems. The observation models are obtained from a combination of INS and GPS measurements.

The computation time of the Kalman filter depends on the dimension n of the integrated system model state vector. The number of computations per recurrence is on the order of n^3. Any reduction in the number of states will benefit directly in computation time. Moreover, the reduction of an integrated system model will bring many benefits to engineering realization. Since most currently available control design methods only work on small dimension systems; the complexity of a higher order model often makes it difficult to obtain a stable system.

The objective of model order reduction is to find a lower order model which preserves the dynamics of more complex high-order systems. In this paper, a 21-states high-order integrated model is investigated first, based on this, a 15-states reduced-order model is developed by eliminating states which are unobservable or weakly observable. A Monte Carlo analysis shows that the reduced-order model presented in this paper provides satisfactory accuracy for aircraft navigation.

DESCRIPTION OF THE INTEGRATED GPS/INS SYSTEM

The Dynamic Error Model

In this paper, an INS platform is considered as a local-level, NEU (north, east, and upper), wander-azimuth system.

A high-order dynamic error model of Integrated GPS/INS can be written as:

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$$\dot{X}(t) = A(t)X(t) + G(t)W(t) \tag{1}$$

where X(t) is derived of the state vector X(t), and X(t) consists of various errors such that

$$\dot{X}(t) = [\phi_x, \phi_y, \phi_z, \delta V_x^P, \delta V_y^P, \delta V_z^P, \delta\varphi, \delta\lambda, \delta h, \delta\alpha,$$
$$\varepsilon_{cx}, \varepsilon_{cy}, \varepsilon_{cz}, \varepsilon_{rx}, \varepsilon_{ry}, \varepsilon_{rz}, \nabla_x, \nabla_y, \nabla_z, \delta_{tu}, \delta_{tru}]^T$$

with

ϕ_x, ϕ_y	is attitude error angles;
ϕ_z	is azimuth error angles;
$\delta\lambda$	is error in east longitude;
$\delta\varphi$	is error in north latitude;
δh	is error in altitude;
δv_x^P	is x-velocity error;
δv_x^P	is y-velocity error;
δv_x^P	is z-velocity error;
$\delta\alpha$	is wander-azimuth angle error;
ε_{cx}	is x-gyro constant drift rate;
ε_{cy}	is y-gyro constant drift rate;
ε_{cz}	is z-gyro constant drift rate;
ε_{rx}	is x-gyro first-order Marov drift;
ε_{ry}	is y-gyro first-order Marov drift;
ε_{rz}	is z-gyro first-order Marov drift;
∇x	is x-accelerometer zero bias;
∇y	is y-accelerometer zero bias;
∇z	is z-accelerometer zero bias;
δ_{tu}, δ_{tru}	is clock bias and clock drift rate, respectively.

The vector of dynamics noise W(t) is given by

$$W(t) = [w_{\varepsilon_{cx}}, w_{\varepsilon_{cy}}, w_{\varepsilon_{cz}}, w_{\varepsilon_{rx}}, w_{\varepsilon_{ry}}, w_{\varepsilon_{rz}},$$
$$w_{\nabla_x}, w_{\nabla_y}, w_{\nabla_z}, w_{\delta_{tu}}, w_{\delta_{tru}}]^T$$

The matrix A(t) (21 by 21) is the integrated system dynamic matrix, it contains eighty-five percent zero elements. The matrix G(t) (21 by 11) is the coefficient matrix, it contains ninety-six percent zero elements. They are sparse matrices.

The Observation Model

Two traditional methods of GPS/INS integration are the tightly coupled and the cascaded approach. In tightly coupled methods, satellite pseudoranges are used as measurements by the integration filter. In the cascaded method, the measurement consists of the navigation solution which is three-dimensional position and velocity. Each of the methods has its own advantages and disadvantages. For a more detailed discussion see [5]. In this example the tightly coupled method is used. In the observation model of integrated GPS/INS, the measurement vectors consist of combinations of the platform INS and GPS. From the platform INS position, the distance ρ_{Ij} between the aircraft and jth satellite can be calculated. Let GPS pseudorange measurements be P_{ji} then the observation equation can be expressed as

$$Z_{\rho j}(t) = \rho_{Ij} - \rho_j \tag{2}$$

where $Z_{pj}(t)$ is the discrepancy between the measurements given by the two systems. Then

$$\rho_j = [(x - x_{sj})^2 + (y - y_{sj})^2 + (z - z_{sj})^2]^{\frac{1}{2}} + \delta\rho_j \tag{3}$$

where (x, y, z) is the true position of the aircraft in the Earth Centered Earth Fixed (ECEF) coordinate system, (X_{sj}, Y_{sj}, Z_{sj}) is the jth satellite position in the ECEF coordinate system, δ_{pj} is the distance error, mainly caused by clock biases:

$$\delta\rho_j = \delta_{tu} + v_j \tag{4}$$

where δ_{iu} is a clock bias, v_j is the GPS receiver measurement noise. Assume that the platform INS position (x_i, y_i, z_i) in the ECEF coordinate system, is transformed from a wander-azimuth system. Then

$$\begin{cases} x_I = x + \delta x_I \\ y_I = y + \delta y_I \\ z_I = z + \delta z_I \end{cases} \tag{5}$$

where (δ_{x1}, δ_{y1}, δ_{z1}) is the positioning error of the platform INS, Pij, is then written as

$$\rho_{Ij} = [(x_I - x_{sj})^2 + (y_I - y_{sj})^2 + (z_I - z_{sj})^2]^{\frac{1}{2}} \tag{6}$$

Eq. (2) linearized form is:

$$\delta Z(t) = H(t)X(t) + V(t) \tag{7}$$

where $\delta_2(4)$ is a correction to the difference between the GPS and INS measurement. H(t) is the integrated system measurement matrix, it contains eighty percent zero elements and is a sparse matrix. V(t) is the GPS receiver measurement noise.

MODEL ORDER REDUCTION METHOD INVESTIGATION

Controllability and Observability

The ideas of system controllability and observability play a key role in the presentation and understanding of model reduction since the stability of the Kalman filter is determined by these two basic characteristics.

The concepts of controllability and observability are embodied in the system controllability and observability grammians. For system (1) and (7), the controllability and observability grammians can be defined as [4]:

The controllability grammian is

$$\overline{P} = \int_0^\infty e^{At} G G^T e^{A^T t} dt \tag{8}$$

The observability grammian is

$$\overline{Q} = \int_0^\infty e^{A^T t} H H^T e^{A t} dt \tag{9}$$

For system (1) and (7), if A(t) has all of its eigenvalues in the open left half of the plane then $\overline{P}$ and $\overline{Q}$ are the unique symmetric positive semi-definite matrices which satisfy the following Lyapunov equation

$$A\overline{P} + \overline{P}A^T + GG^T = 0 \tag{10}$$

$$A^T\overline{Q} + \overline{Q}A + H^T H = 0 \tag{11}$$

The connection of system controllability and observability to $\overline{P}$ and $\overline{Q}$ is said to be [4]:

a) $\overline{P}$ is positive definite if, and only if, (A, G) is completely controllable;

b) $\overline{Q}$ is positive definite if, and only if, (A, H) is completely observable.

Mathematically, the controllability and observability of the system can be dealt with using the same method [1]. Here we consider observability. The observability of the system can usually be determined by examining the observability matrix M [1]:

$$M = \begin{bmatrix} H(0) \\ H(1)A(0) \\ \dots \\ H(n)A(n)A(n-1)\dots A(0) \end{bmatrix} \tag{12}$$

If M is a full-rank matrix, system (1) and (7) are observable. Otherwise they are unobservable.

It is not easy to examine observability in systems (l) and (7) because of the difficulty in determining the rank of matrix M. Instead of determining the rank of matrix M, the observability of system (1) and (7) can be determined by computing the singular values of the matrix M.

Consider singular value decomposition (SVD) of the matrix M such that

$$M = U\Sigma V^T =$$

$$U \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & \ddots & \\ & & & & & \sigma_n \end{bmatrix} V^T$$

where $\quad U^T U = I, V^T V = I$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq \cdots \geq \sigma_n \geq 0$$

are the singular values of the matrix M. If the number of nonzero singular values is equal to the dimension n, the systems (1) and (7) are observable; however, if the number of singular values is less than the dimension n, the system is unobservable.

Since matrices A(t) and H(t) have special properties (sparse), an approach called the classical manner of exploiting sparsity can be employed so as to carry out the singular value decomposition of matrix M. A more detailed description of this approach is found in [7].

The above method of examining the observability of integrated systems is workable theoretically, but not suggested in practice because of the complex calculation. A rough approach is used to examine the observability of integrated systems, that is, studying the possibility of estimating the state from the output. If a dynamic system is observable, all the states of the dynamic equation can be estimated at the output. Contrarily, if a state cannot be estimated from the output, it is unobservable.

Reduced Order Model for Integrated GPS/INS

Model order reduction methods have been presented in numerous research papers. These methods can be divided into two groups. The first group attempts to retain the dominant modes of the original system, such as Aggregation methods, Modal methods, Lyapunov function methods and Perturbation methods [6]. Another approach is based on applying an identification procedure to input-output data obtained by driving the original system with a specific input. For integrated GPS/INS systems, since the methods of model order reduction are applied in engineering practice, we are interested in the first group of methods, that is, to eliminate some states in order to reduce computational difficulties.

An approach to eliminating states is based on the link between the physical system and its mathematical models. According to [2], those states which are unobservable contribute little to the response of the system. Thus, these states may be eliminated while still preserving the basic performance of the dynamics system.

For integrated systems (1) and (7), we can determine which states are observable and which are unobservable using a Kalman filter. Gyros constant drift, accelerometers zero bias and user clock and clock drift bias states cannot be observed by the Kalman filter. Thus, in system (1), three states of gyros constant drift are removed, and three states of accelerometers zero bias, which are considered white noise, are eliminated. Since the use of the clock and clock drift bias states are required in the tightly coupled case, they cannot be eliminated and are treated as consider states increasing the filter's robustness [5]. A reduced-order dynamic of the integrated GPS/INS can be written as:

$$\dot{X}_R(t) = A_R(t)X_R(t) + G_R(t)W(t) \tag{13}$$

$$X_R(t) = [\phi_x, \phi_y, \phi_z, \delta V_x^p, \delta V_y^p, \delta V_z^p, \delta\varphi, \delta\lambda, \delta h, \delta\alpha,$$
$$\varepsilon_{rx}, \varepsilon_{ry}, \varepsilon_{rz}, \delta_{lu}, \delta_{lru}]^T$$

The matrix $A_R(t)$(15 by 15) is the reduced-order integrated system dynamic matrix, the matrix $G_R(t)$(15 by 11) is the coefficient matrix. The observation equation can be written as:

$$\delta Z_R(t) = H_R(t)X_R(t) + V(t) \tag{14}$$

where $\delta Z_R(t)$ is a correction to the difference between GPS and INS measurements, $H_R(t)$ is the reduced order integrated system measurement matrix, V(t) is the measurement noise.

A FASTER TRACKING APPROACH FOR KALMAN FILTERING

Consider the discrete form of the above reduced order integrated GPS/INS system as:

$$X(k+1) = \Phi(k) \cdot X(k) + \Gamma(k) \cdot W(k) \tag{15}$$

$$Z(k) = H(k) \cdot X(k) + V(k) \tag{16}$$

where Φ < (k) is a state transition matrix, γ(k) is the coefficient matrix, H(k) is the measurement matrix, Φ(k), γ(k) and H(k) are sparse matrices. w(k) and v(k) are white noise with associated variance matrices Q(k) and R(k), respectively.

A Kalman filtering algorithm with closed loop control is employed. The algorithm is summarised as

$$P(k/k-1) = \Phi(k)P(k-1/k-1)\Phi^T + \Gamma(k)Q(k)\Gamma^T(k) \tag{17}$$

$$K(k) = P(k/k-1)H^T(k)[H(k)P(k/k-1)H^T(k) + R(k)]^{-1} \tag{18}$$

$$P(k/k) = [I - K(k)H(k)]P(k/k-1)[I - K(k)H(k)]^T$$
$$K(k)R(k)K^T(k) \tag{19}$$

Since the state vectors of integrated systems are error elements, the control aim is to eliminate these errors so a direct control law can be considered as

$$U(k) = -\hat{X}(k/k) \tag{20}$$

State estimate vector can be expressed as:

$$\hat{X}(k/k) = K(k)Z(k) \tag{21}$$

Table 1. Navigation Errors Comparing High-Order Model (21 States) with Reduced-Order Model (15 States)

Navigation error	High order model(1σ)	Reduced order model(1σ)
Pitch angle error(arcsec)	50.61	59.06
Roll angle error(arcsec)	54.17	70.15
Azimuth angle error(arcsec)	160.28	240.92
X-velocity error(m/s)	0.08	0.10
Y-velocity error(m/s)	0.09	0.10
Z-velocity error(m/s)	0.08	0.13
Latitude error(m)	8.05	9.31
longitude error(m)	8.08	9.10
Height error(m)	8.40	9.54

$$\hat{X}(k/k-1) = 0 \tag{22}$$

where P(k/k) and P(k/k-1) are update covariance matrices and predicted covariance matrices, respectively. $\hat{X}(k/k)$ and $\hat{X}(k/k-1)$ are updated state vector and predicated stare vector, respectively. K(k) is gain matrix.

For the above Kalman filter algorithm, most of the computation time concentrates on $\Phi(k)P(k/k)\Phi^T(k)$ in Equation (17) [3]. Since state transition Φ(k) is a sparse matrix, that is, approximately seventy-five percent of the elements of the matrix are zero. If the multiplication with non-zero elements need be performed, this reduces the number of multiplications. It is suggested individual elements be used for the Kalman filter, instead of treating matrix Φ(k). Thus a large computational savings can be obtained. For $\Phi(k)P(k/k)\Phi^T(k)$ calculations, which require $2n^3 = 2 \times 15^3 = 6750$ multiplications, only $L^2 - J^2 = 55^2 - 8^2 = 2961$ multiplications are now required, where L and J denote the number of non-zero elements and 1 elements in the state transition matrix, respectively.

SIMULATION

To investigate the behaviour of the reduced-order integrated system as compared to the

Table 2. Computation Time Comparison of High-Order and Reduced-Order Model without Faster Tracking Approach

Integrated system state	Computation time in each step
21-states	0.15 s
15-states	0.06 s

Table 3. Computation Time Comparison of 15 States Reduced Model with and without Faster Tracking Kalman Filter

Faster Tracking Approach for Kalman Filter	Computation Time in Each Step
No	0.06 s
Yes	0.02 s

high-order system, a Monte Carlo simulation test is carried out.

An aircraft trajectory is designed for actual and complex high-dynamic aircraft flight. It is assumed that an aircraft takes off, flies, and maneuvers for a total time 5300 seconds. The maneuvers include climbing, pitching, rolling, and turning.

Table 1, previous page, shows aircraft navigation error accuracy using high-order and reduced-order integrated models. Table 2 shows computation time comparing the high-order with reduced-order integrated model. Table 3 shows computation of reduced-integrated model with and without faster tracking approach.

Figures 1-3 show navigation error curves of high-order and reduced-order integrated GPS/INS, respectively. A comparison of Figures 1 and 2 show no difference in position error using high-order and reduced-order integrated models. There is a small difference in attitude error angles ϕ_x, ϕ_y and velocity error div δV_x^P, δV_x^P, but the big difference in azimuth error angle ϕ_z and z-velocity error occur using high-order and reduced-order models. This is caused by simplifying the gyro and accelerometer models.

CONCLUSION

Based on above simulation results, the following conclusions can be drawn:

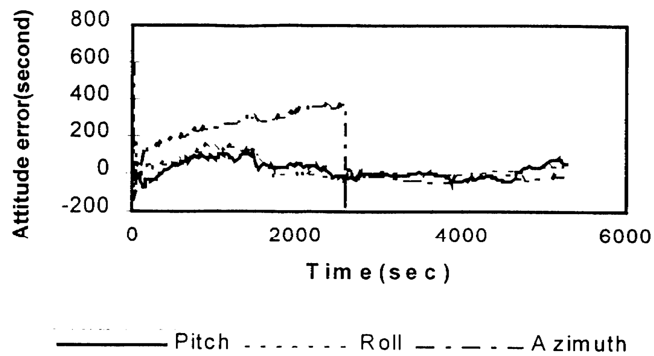

Fig. 1A. Aircraft Attitude Error Curves (21 States)

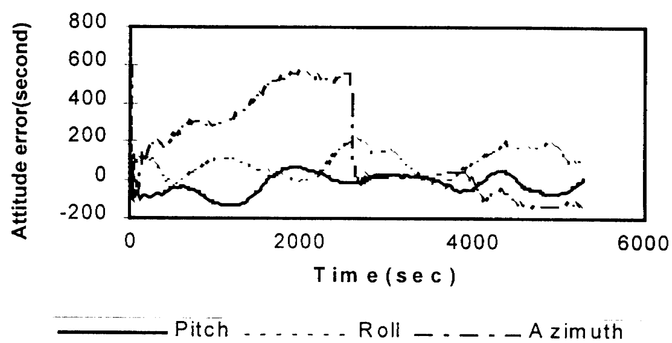

Fig. 1B. Aircraft Attitude Error Curves (15 States)

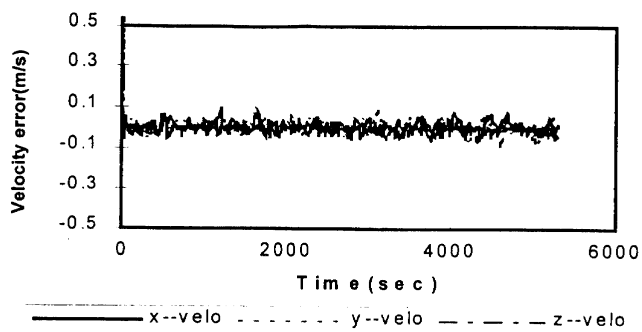

Fig. 2A. Aircraft Velocity Error Curves (21 States)

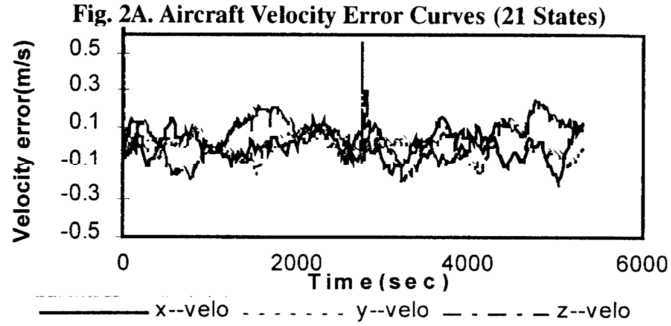

Fig. 2B. Aircraft Velocity Error Curves (15 States)

1) Reduced-order integrated models can provide satisfactory accuracy for aircraft navigation.

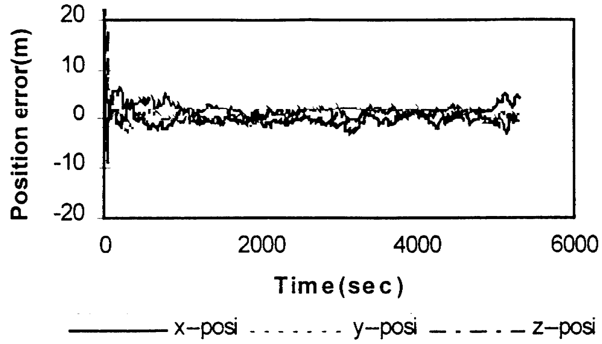

Fig. 3A. Aircraft Position Error Curves (21 States)

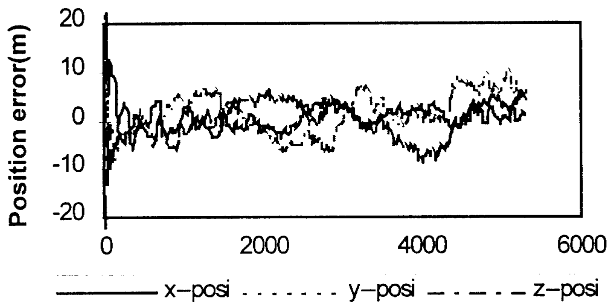

Fig. 3B. Aircraft Position Error Curves (15 States)

2) The computation time of the Kalman filter is proportional to the cube of the number of integrated system state vector. The time reduction was approximately 60% by the model reduction from 21 Order to 15 Order, which brings benefit to engineering realization. The faster tracking approach for Kalman filters presented in this paper is efficient for the integrated system, it reduced computation time approx. 67%.

3) The constant drift rate of gyros is removed in reduced-order integrated systems, in fact, the constant drift rate of gyros can be measured and precompensated in order to further improve navigation accuracy.

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Mr. Xiufeng He
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Dear Mr. :

Thank you for letting us publish your fine article, "A Reduced-Order Model for Integrated GPS/INS," in the March, 1998 issue of the IEEE *Aerospace and Electronic Systems Magazine*.

I now enclose a copy of this March issue in which your article starts on Page 40.

Incidentally, the Institute for Scientific Information evaluated 6000 of the world's top journals. I don't know how many fell into the category of "Aerospace Engineering and Technology." However, thanks to authors like you, our *Aerospace and Electronic Systems Magazine* came out second from the top in this category!

We hope that you will let us publish more of your interesting work as it progresses.

Very truly yours,

Henry Oman

Henry Oman, Editor-in-Chief