An Operational Realization of a Global Vertical Datum for MSL Studies Using VLBI Baselines and Its Stability

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Purpose

- Establishment of a global vertical datum using VLBI observations
- Evolution of the vertical motion of the reference stations of the vertical datum
- Stability and nature of the vertical station motions

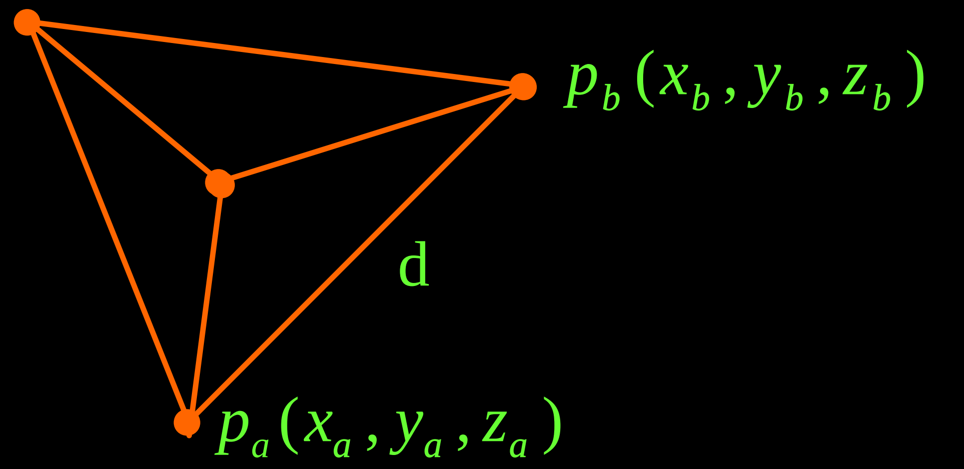
Approach

- *Global vertical datum realizations* via VLBI tetrahedron coordinates calculated using MINOLESS
- Tetrahedral deformations using *finite element* approach
- Station height stability – the effect of plate motions
- Statistical properties of *height differences* from epoch to epoch via turning point test and correlogram of station height differences

Fundamental figures

1-D → baseline	→	two stations
2-D → triangle	→	three stations
3-D → tetrahedron	→	four stations.

Mathematical model



$$d_{ab}^2 = (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2$$

Statistical model

$$y = F(\mathbf{x}) + u$$

$$u \sim (0, \Sigma_u)$$

- *Y* is the *nx1* vector of observed baseline lengths
- *u is* the *nx1* vector of disturbances with zero mean and a positive definite variance/covariance matrix
- $F(\mathbf{x})$ is the nonlinear function of distance
- **x** is the *mx1* vector of unknown station coordinates = no. of stations x 3)

Linearized model

$$d_{ab_{Obs}} = d_{0ab} + \left.\frac{\partial d_{ab}}{\partial x_a}\right|_0 (x_a - x_{0a}) + \left.\frac{\partial d_{ab}}{\partial y_a}\right|_0 (y_a - y_{0a}) + \left.\frac{\partial d_{ab}}{\partial z_a}\right|_0 (z_a - z_{0a}) +$$

$$\left.\frac{\partial d_{ab}}{\partial x_b}\right|_0 (x_b - x_{0b}) + \left.\frac{\partial d_{ab}}{\partial y_b}\right|_0 (y_b - y_{0b}) + \left.\frac{\partial d_{ab}}{\partial z_b}\right|_0 (z_b - z_{0b}) + u$$

For a tetrahedron:

6 baselines ➔ *y* (6x1)➔ 6 observation equations
4 stations ➔ 12 unknown corrections to the nominal coordinates
+ 6 unknown observation errors

Linearized model (cont.)

Constructing normal equations through least squares formalism reduces the number of unknowns to 12.

$$? \mathbf{y} = \mathbf{A}\Delta\mathbf{x} + \mathbf{u}$$

where,

$? \mathbf{y}_{6x1} := d_{\text{obs}} - d_{\text{nominal}}$

$\mathbf{A}_{6x12}$ coefficient (design) matrix - from partials

$? \mathbf{x}_{12x1} := \mathbf{x} - \mathbf{x}_{\text{nominal}}$

Normal equations and MINOLESS

$$(\mathbf{A}^{\mathrm{T}}\mathbf{S}_u^{-1}\mathbf{A})\mathbf{?}x = \mathbf{A}^{\mathrm{T}}\mathbf{S}_u^{-1}\mathbf{?}\mathbf{y}$$

$$R(\mathbf{A}^{\mathrm{T}}\mathbf{S}_u^{-1}\mathbf{A}) = 6 \qquad \dim(\mathbf{A}^{\mathrm{T}}\mathbf{S}_u^{-1}\mathbf{A}) = 12$$

➔ there are multitude of solutions that satisfy the above system. We choose the following solution (MINOLESS)

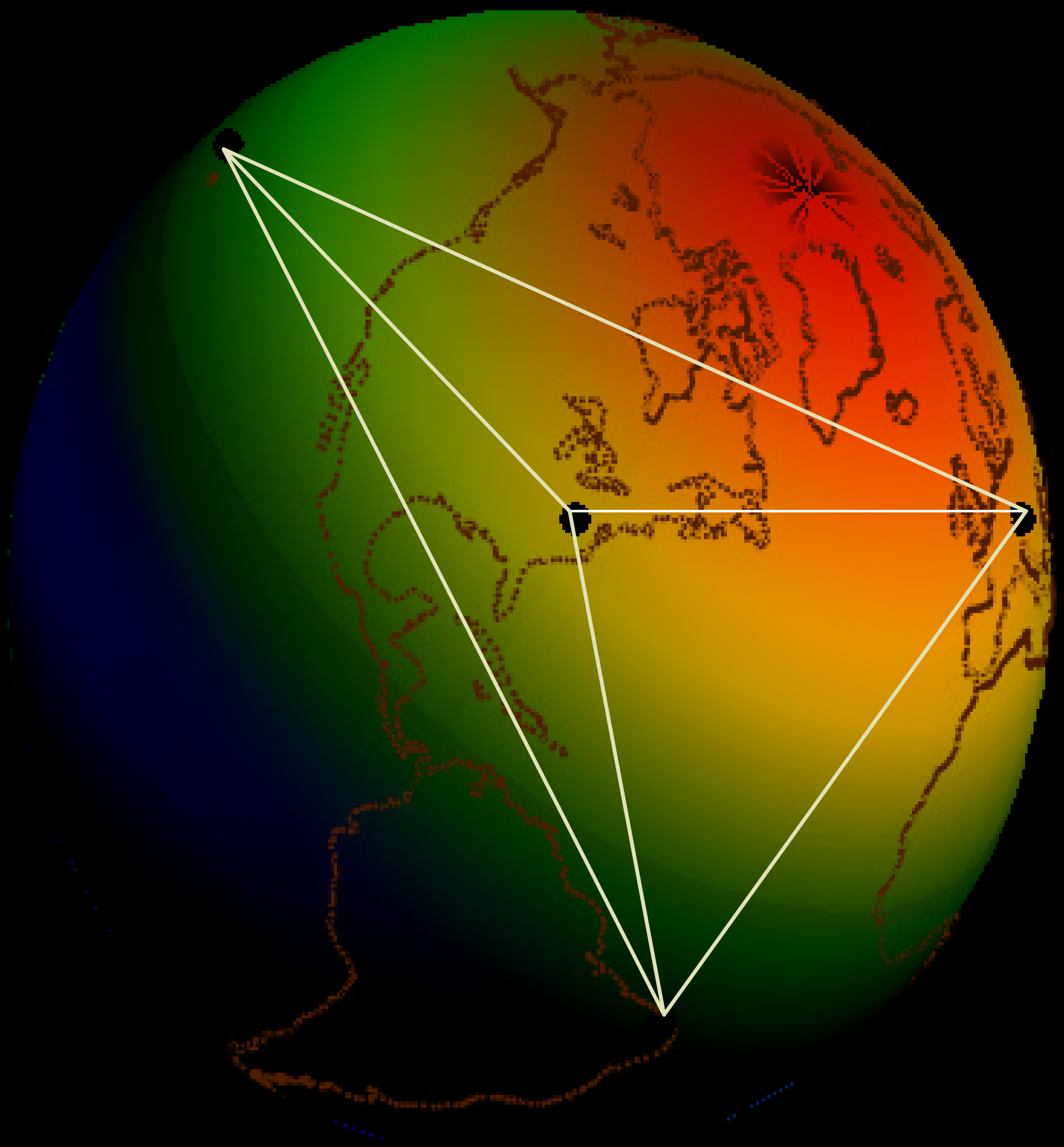
$$\mathbf{?}x = (\mathbf{A}^{\mathrm{T}}\mathbf{S}_u^{-1}\mathbf{A})^{+}\mathbf{A}^{\mathrm{T}}\mathbf{S}_u^{-1}\mathbf{?}\mathbf{y}$$

Solution properties

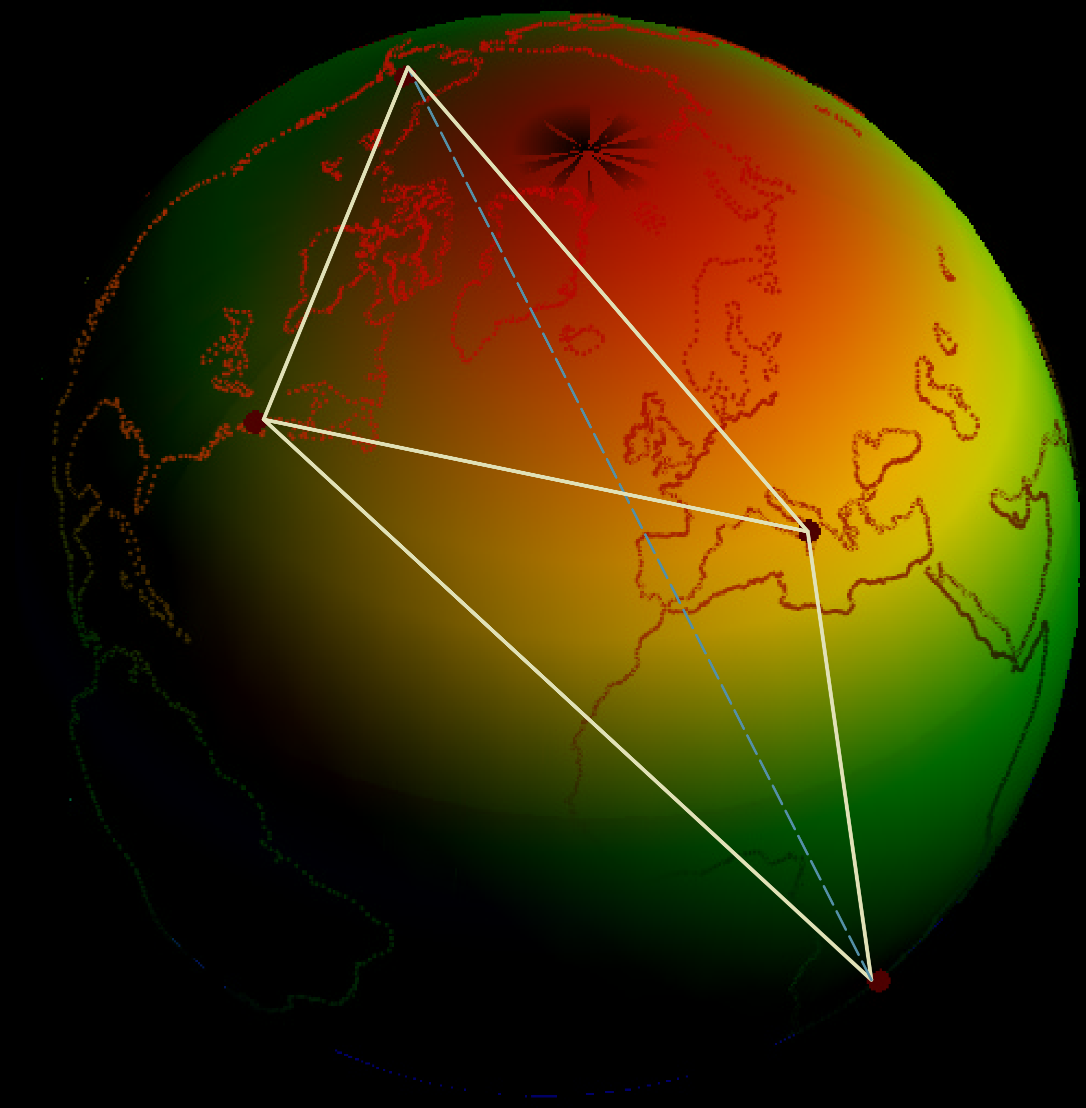
- Minimum variance solution, $trD(\hat{x}) = \min$
- Minimum norm solution, $|\Delta x| = \min$ (*Auf felderung*, Wolf, 1973)
- The corrections to the nominal coordinates of the center of mass of the tetrahedron are zero → the origin is fixed
- The rotation of the tetrahedron to be estimated with respect to the tetrahedron defined by the nominal coordinates is zero (Koch, 1987). Also known as *no net rotation* solution

The above solution is iterated until there are no changes in the baseline lengths. Note that in this particular case, the input baseline lengths remain invariant because there are no redundancies in the figure for an adjustment.

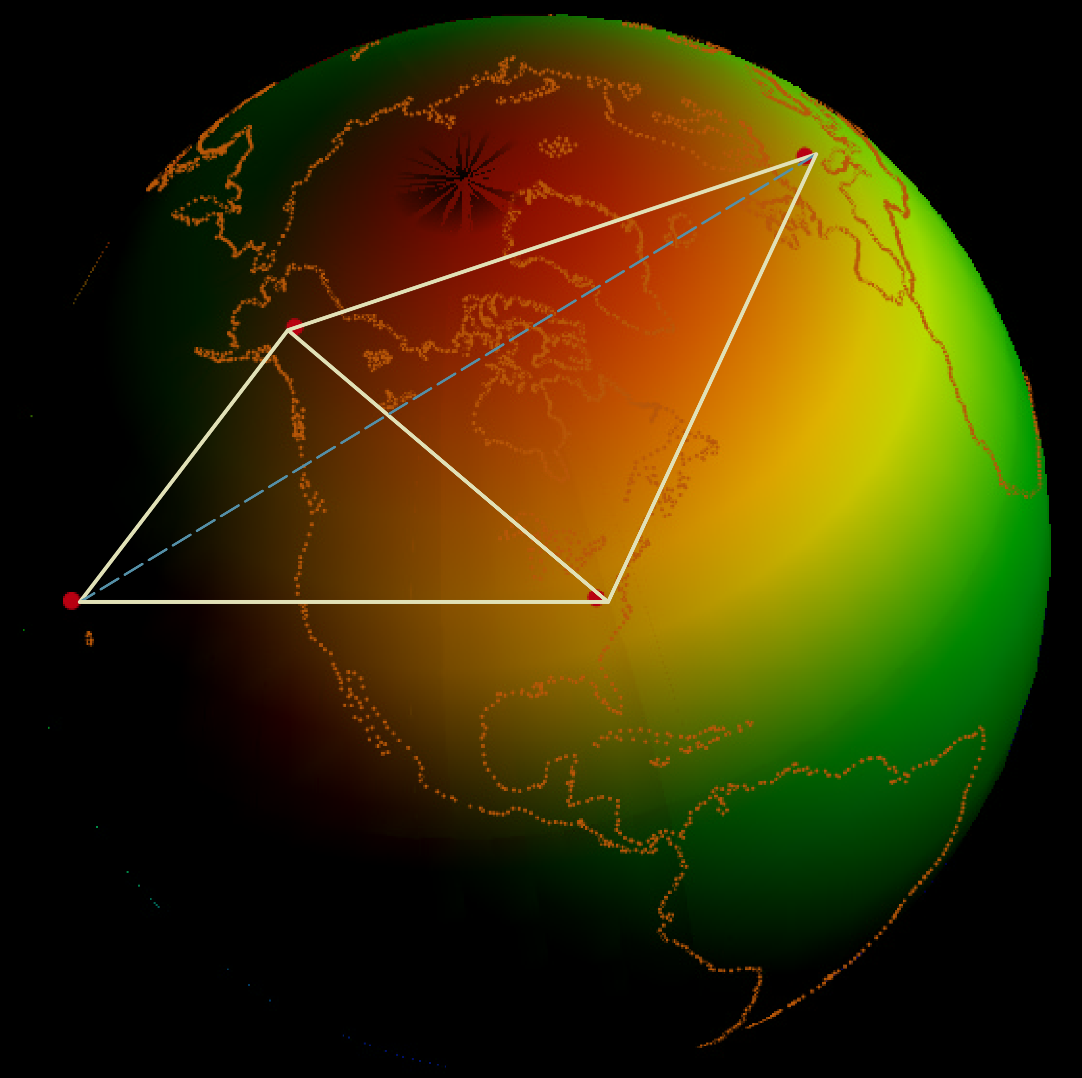
Network: FKNW (Fortleza, Kokee, NRAO20, Wettzell)



GHMW (Gilcreek, Hartrao, Matera, Westford)



GKNW (Gilcreek, Kokee, NRAO20, Wettzell).



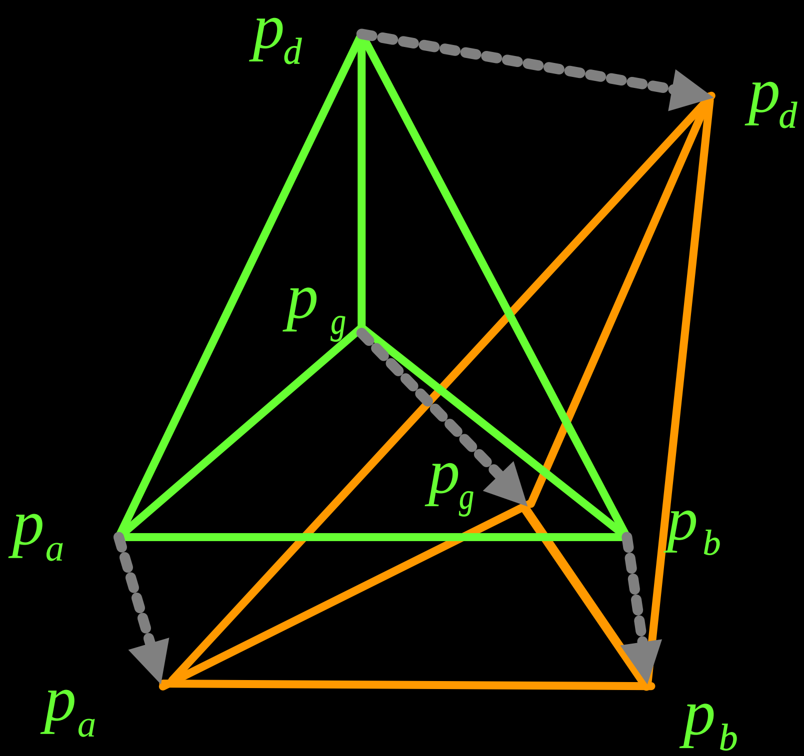
Time series evolution of station coordinates

- *Simultaneous* BL observations are from USNO 1999-3 solution (not corrected for plate motions)
- Nominal coordinates of the stations are defined by the above solution coordinates at the beginning epoch of each tetrahedron baseline observations
- Solutions include the variance/covariance matrices of the baseline lengths
- ➔ We calculate a series of coordinates for each station of three tetrahedra

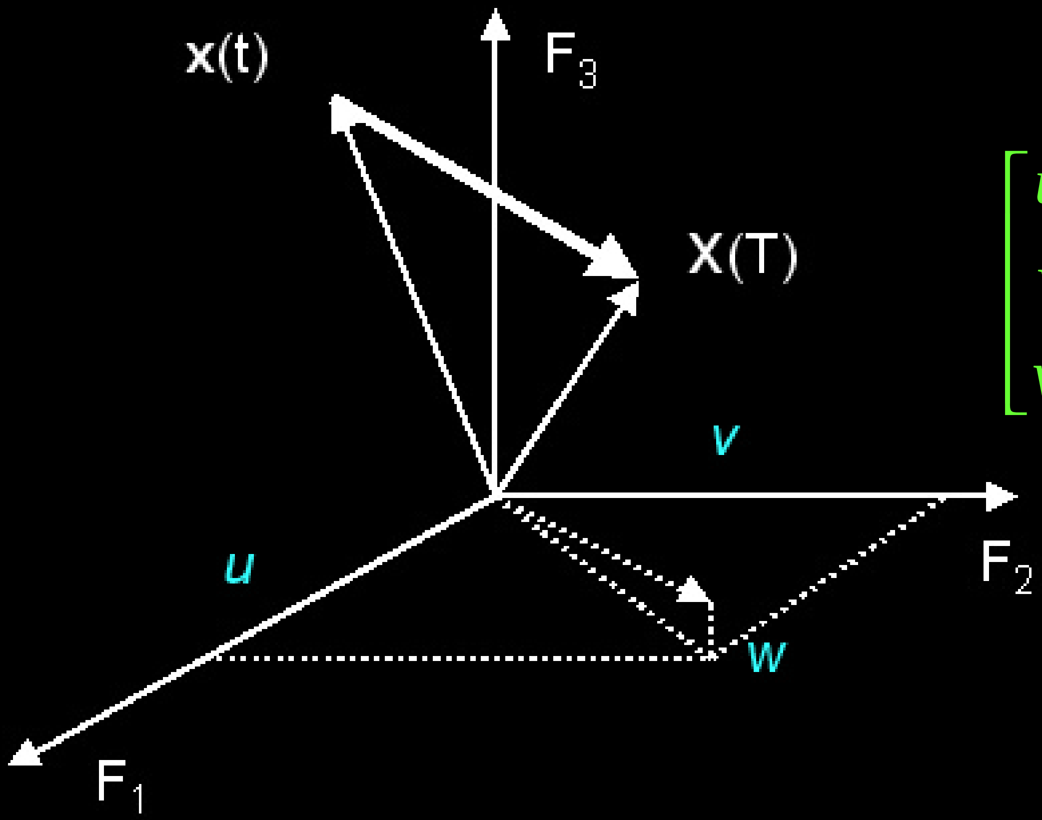
VLBI baseline evolution using strain tensor elements

- Infinitesimal deformations (Grafarend, 1985)
- Strain tensor elements
- Cumulative deformation of the fundamental tetrahedron

Deformation of the fundamental tetrahedron



Displacement vector



$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} := \begin{bmatrix} X(T) - x(t) \\ Y(T) - y(t) \\ Z(T) - z(t) \end{bmatrix}$$

Approximating with a Taylor point (chosen to be the mass or geometric center of the tetrahedron)

$$
\begin{aligned}
u(x,y,z) = u(x_0,y_0,z_0) + u_x(x_0,y_0,z_0)(x-x_0) \\
+ u_y(x_0,y_0,z_0)(y-y_0) \\
+ u_z(x_0,y_0,z_0)(z-z_0) \\
v(x,y,z) = v(x_0,y_0,z_0) + v_x(x_0,y_0,z_0)(x-x_0) \\
+ v_y(x_0,y_0,z_0)(y-y_0) \\
+ v_z(x_0,y_0,z_0)(z-z_0) \\
w(x,y,z) = w(x_0,y_0,z_0) + w_x(x_0,y_0,z_0)(x-x_0) \\
+ w_y(x_0,y_0,z_0)(y-y_0) \\
+ w_z(x_0,y_0,z_0)(z-z_0)
\end{aligned}
$$

Where the mass center of the tetrahedron is given by

$$x_0 = \frac{1}{4}(x_a + x_b + x_d + x_g)$$

$$y_0 = \frac{1}{4}(y_a + y_b + y_d + y_g)$$

$$z_0 = \frac{1}{4}(z_a + z_b + z_d + z_g)$$

Solution

$$\begin{bmatrix} u_a & v_a & w_a \\ u_b & v_b & w_b \\ u_g & v_g & w_g \\ u_d & v_d & w_d \end{bmatrix} = \begin{bmatrix} 1 & x_a - x_0 & y_a - y_0 & z_a - z_0 \\ 1 & x_b - x_0 & y_b - y_0 & z_b - z_0 \\ 1 & x_g - x_0 & y_g - y_0 & z_g - z_0 \\ 1 & x_d - x_0 & y_d - y_0 & z_d - z_0 \end{bmatrix} \begin{bmatrix} u_{a\,x} & v_{a\,y} & w_{a\,z} \\ u_{b\,x} & v_{b\,y} & w_{b\,z} \\ u_{g\,x} & v_{g\,y} & w_{g\,z} \\ u_{d\,x} & v_{d\,y} & w_{d\,z} \end{bmatrix}$$

$U = MA$

If V denotes the volume of the tetrahedron then,

$|M| = 6\,V$

Strain tensor elements

$$e_{ij} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} \quad w_{ij} = \begin{bmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{bmatrix}$$

9 elements for strain tensor and rotations plus three more for the translation

$$u_0 \text{(along x axis)}, v_0 \text{(along y axis)}, w_0 \text{(along z axis)}$$

for a total of 12:

Infinitesimal deformations of the fundamental tetrahedron

Normal strain

$$e_{11} = u_x = \frac{\partial u}{\partial x} \quad e_{22} = v_y = \frac{\partial v}{\partial y} \quad e_{33} = w_z = \frac{\partial w}{\partial z}$$

Shear

$$e_{12} = \frac{1}{2}(u_y + v_x) \quad e_{13} = \frac{1}{2}(u_z + w_x) \quad e_{23} = \frac{1}{2}(v_z + w_y)$$

Infinitesimal Deformations of the Fundamental Tetrahedron

Rigid body motions:

Rotation

$$w_{12} = \frac{1}{2}(u_y - v_x) \quad w_{13} = \frac{1}{2}(u_z - w_x) \quad w_{23} = \frac{1}{2}(v_z - w_y)$$

Translation

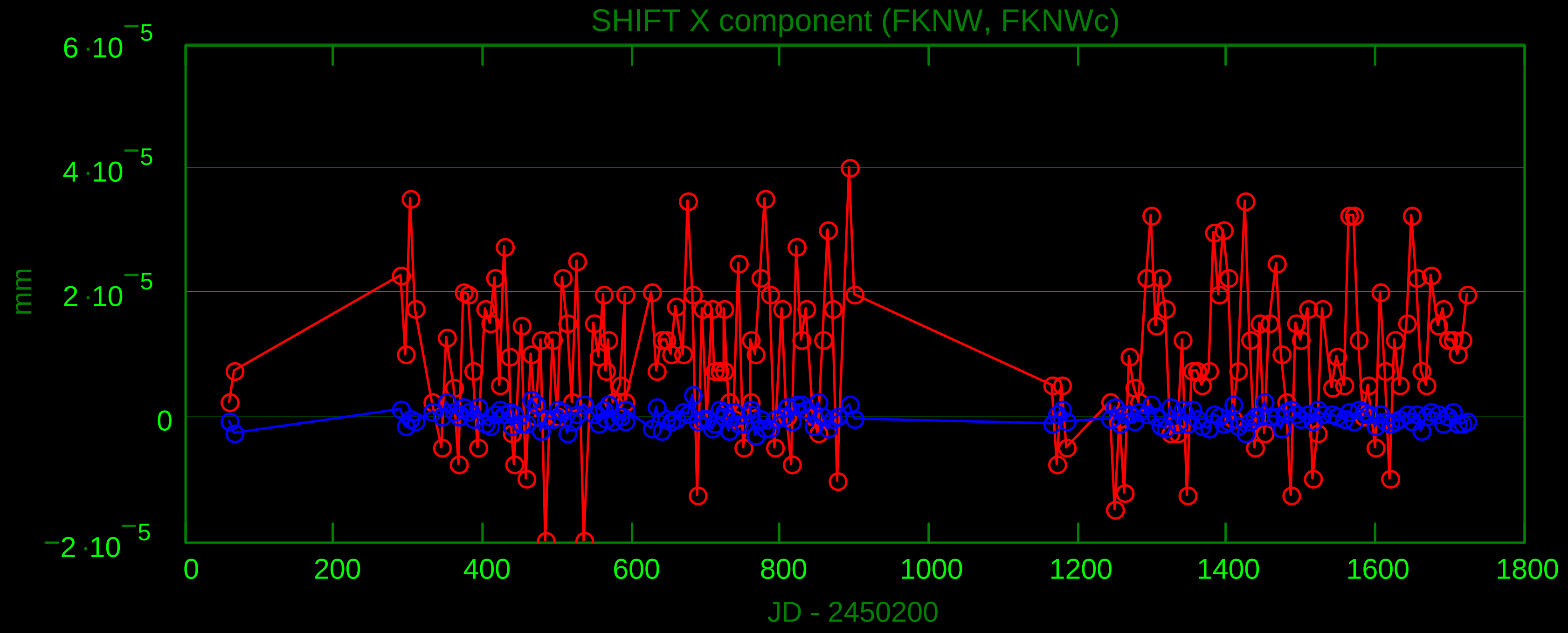
$$u_0 = u(x_0, y_0, z_0) \quad v_0 = v(x_0, y_0, z_0) \quad w_0 = w(x_0, y_0, z_0)$$

Invariant (estimable) quantities

Note that the use of only baseline measurements with no other information implies that the rigid body motion of the tetrahedrons, their translations and individual strain tensor elements are not *invariant* (except certain linear combinations, such as volumetric changes – via normal strain tensor elements, etc…). On the other hand, if the nominal coordinates of the stations are kept the same from epoch to epoch, deformations of the tetrahedron can be traced *consistently* over the time span of the data with respect to the reference tetrahedron.

Numerical Results (sampled)

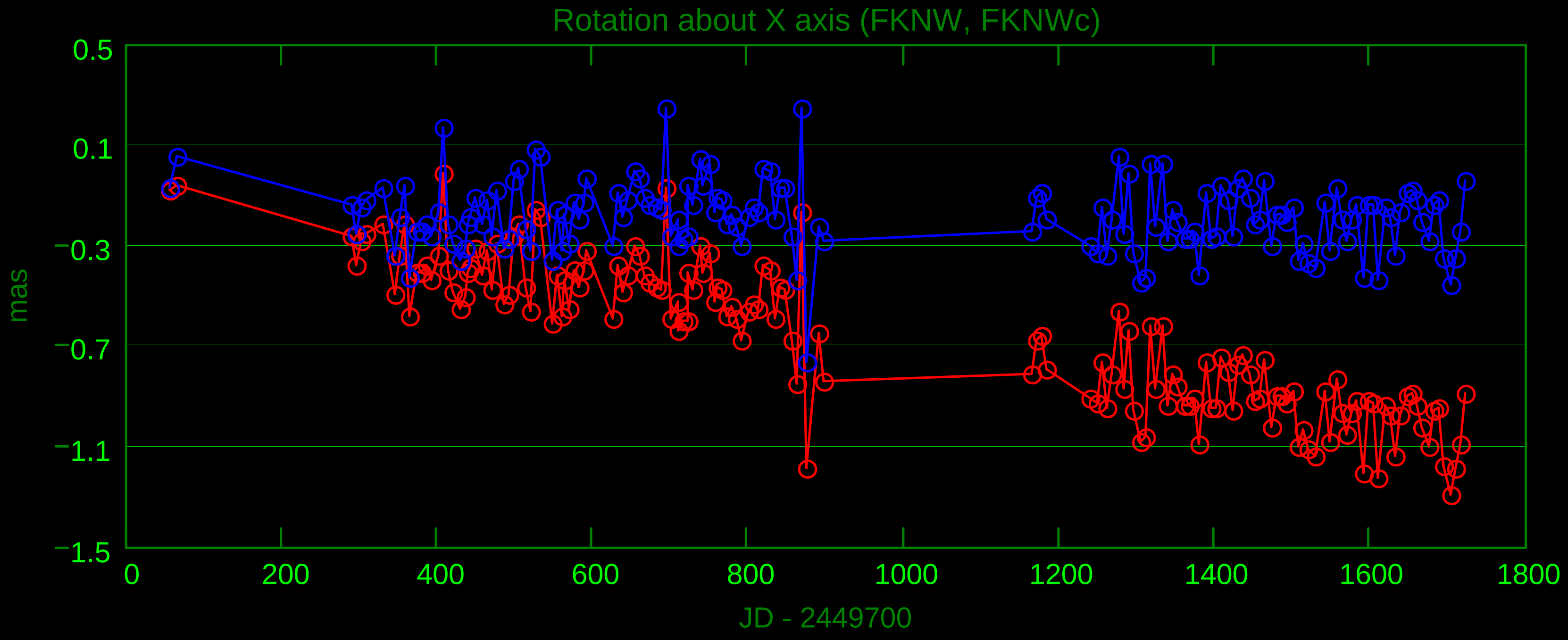
FKNW (shift x)



SHIFT X component (FKNW, FKNWc)
mm
6 ·10⁻⁵
4 ·10⁻⁵
2 ·10⁻⁵
0
−2 ·10⁻⁵
0
200
400
600
800
1000
1200
1400
1600
1800
JD - 2450200

FKNW (rotation about x axis)

-

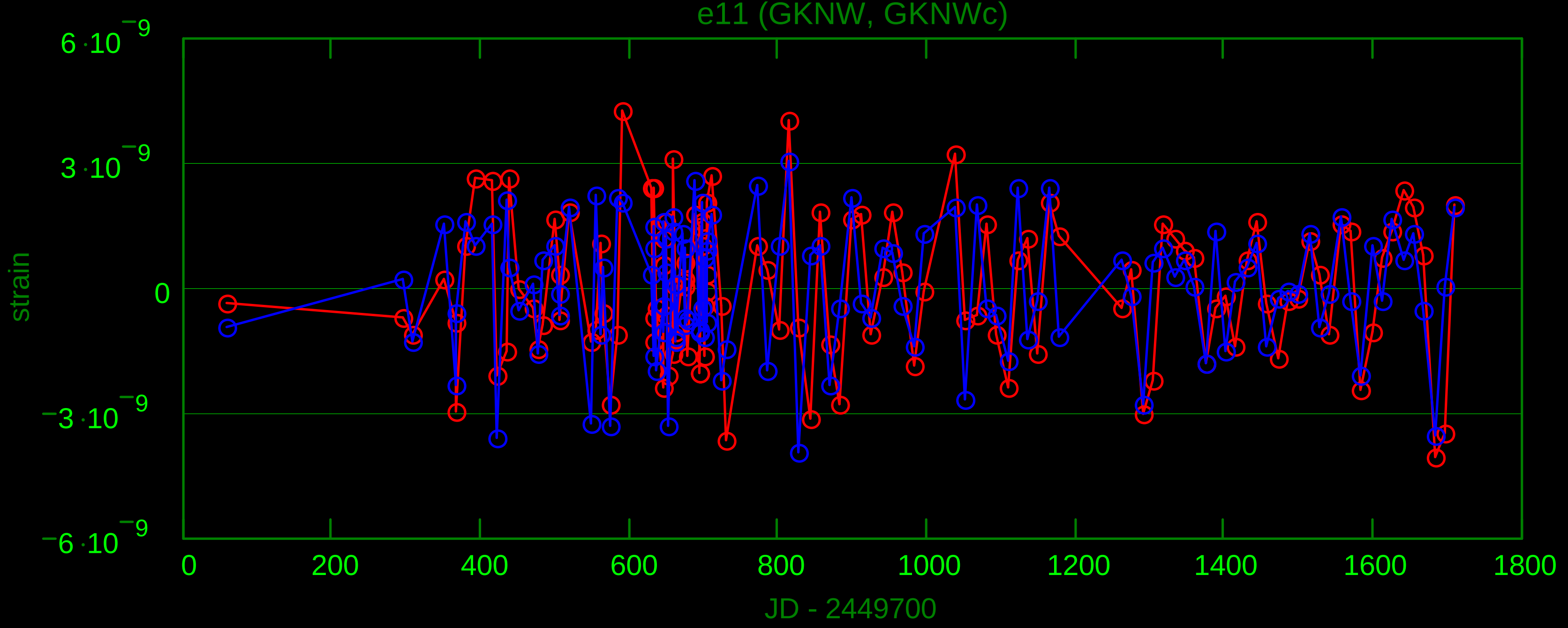


slope = -0.20 mas/yr

slope = -0.02 mas/yr (corrected for PM)

GKNW (normal strain, e_{11})

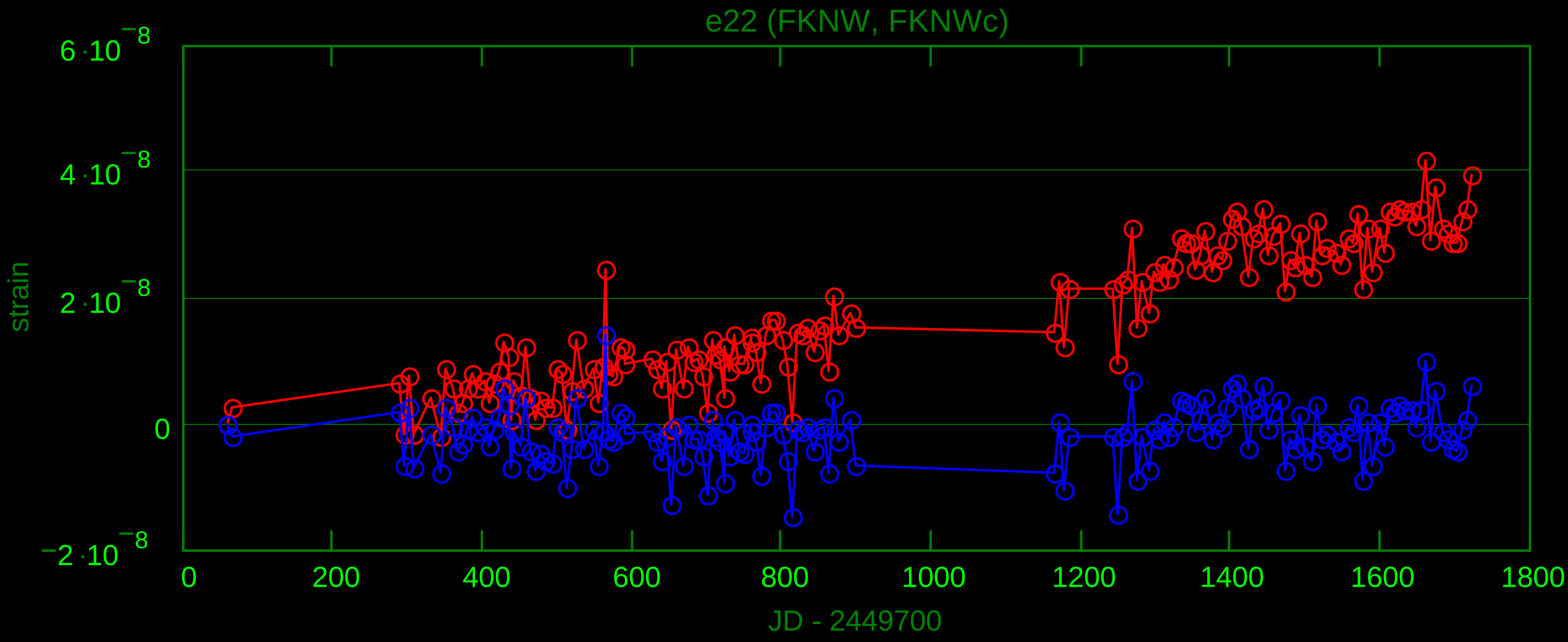
-



mean = -1.9E-11, σ = 1.69E-9

mean = -4.1E-12, σ = 1.54E-9 (corrected for PM)

FKNW (normal strain, e_{22})

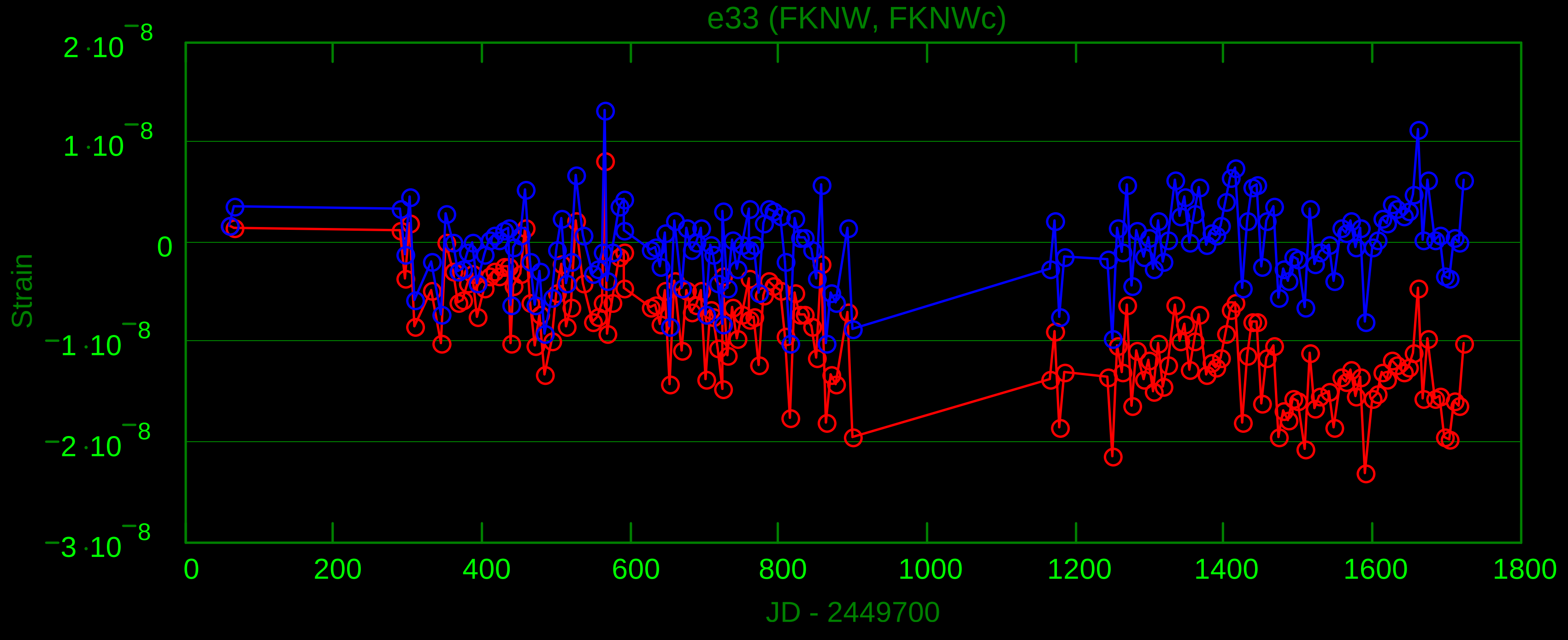


slope = 7.90E-09 strain/yr

slope = 6.71E-10 strain/yr (corrected for PM)

FKNW (normal strain, e_{33})

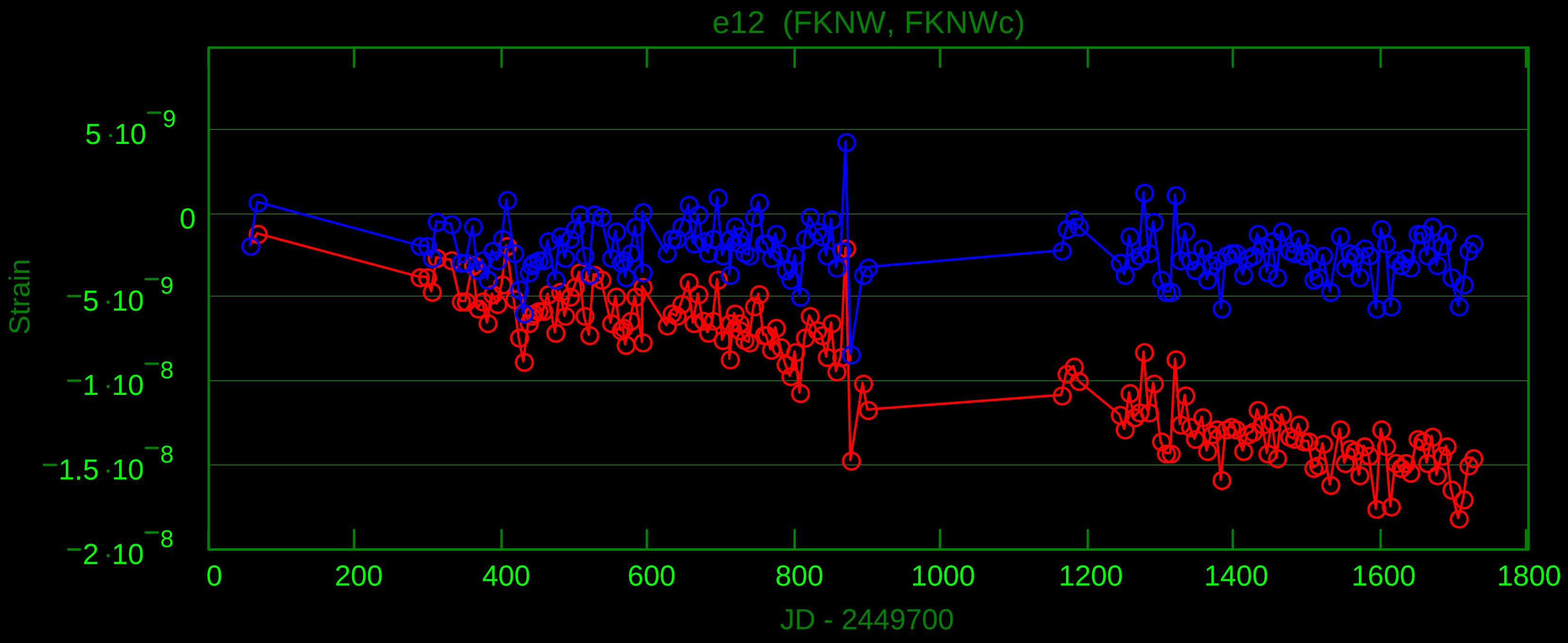
-



slope = -3.00E-09 strain/yr
slope = 5.70E-10 strain/yr (corrected for PM)

FKNW (shear, e_{12})

-

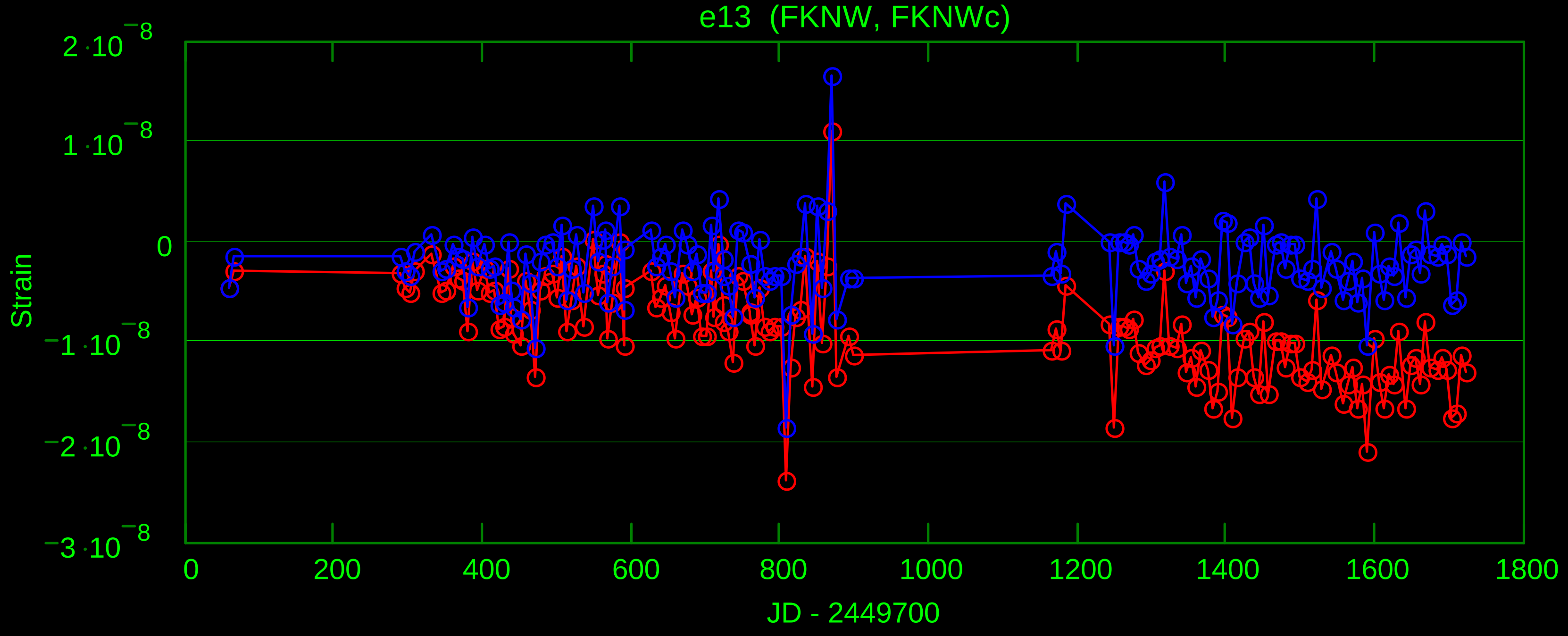


slope = -3.03E-09 strain/yr

slope = -2.33E-10 strain/yr (corrected for PM)

FKNW (shear, e_{13})

-

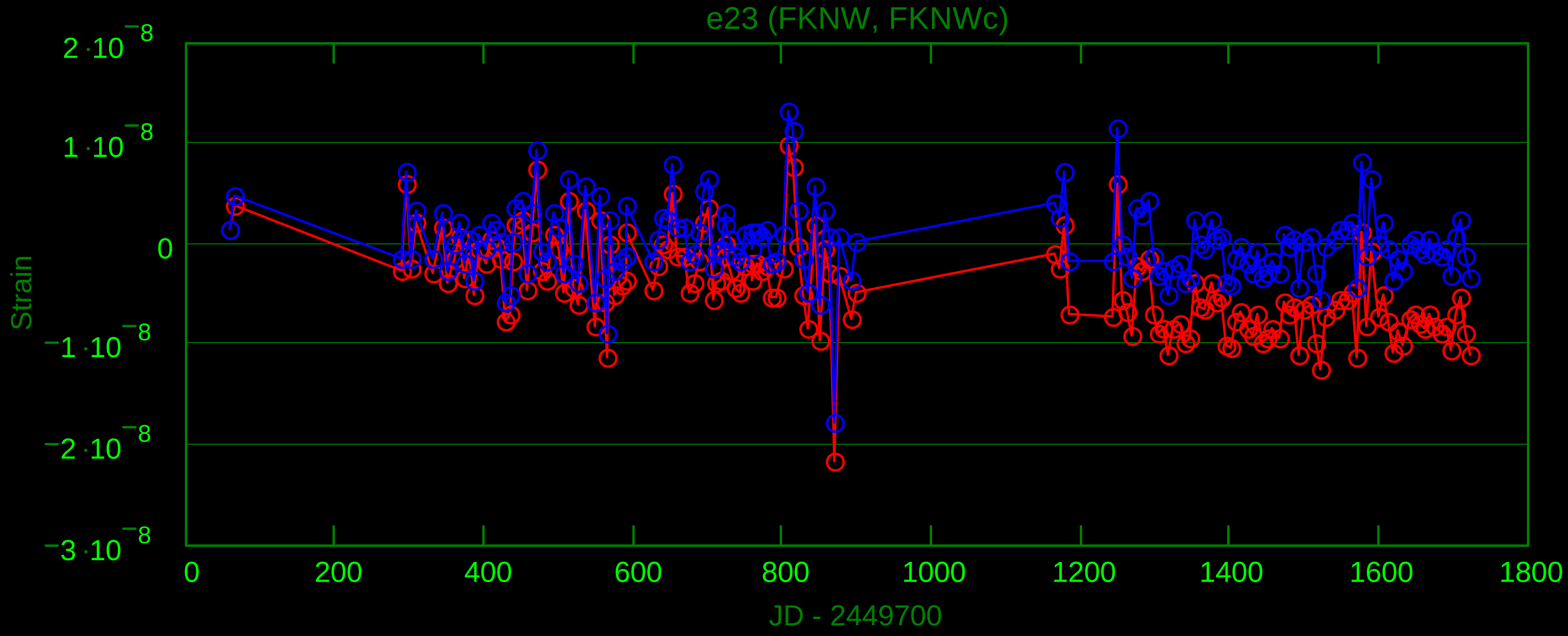


slope = -2.47E-09 strain/yr

slope = 2.99E-11 strain/yr (corrected for PM)

FKNW (shear, e_{23})

-



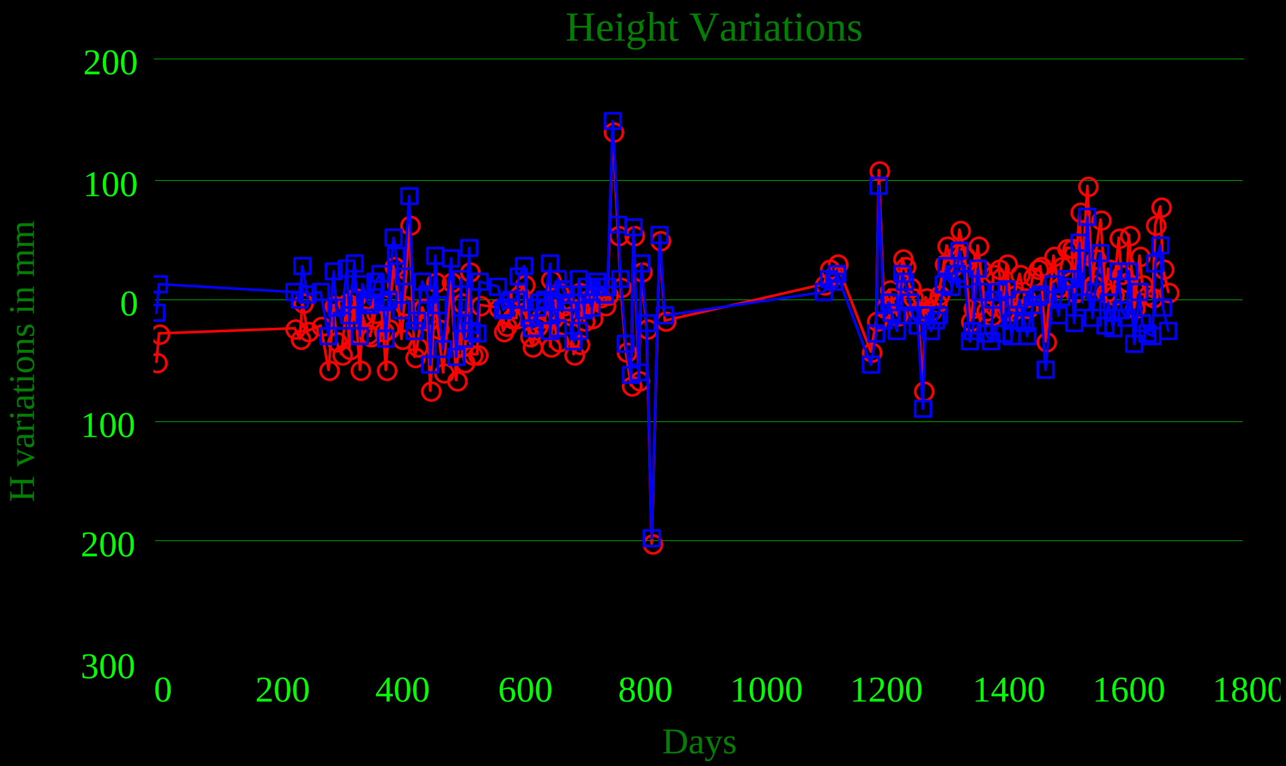
slope = -2.10E-09 strain/yr

slope = -4.11E-10 strain/yr (corrected for PM)

Height variations

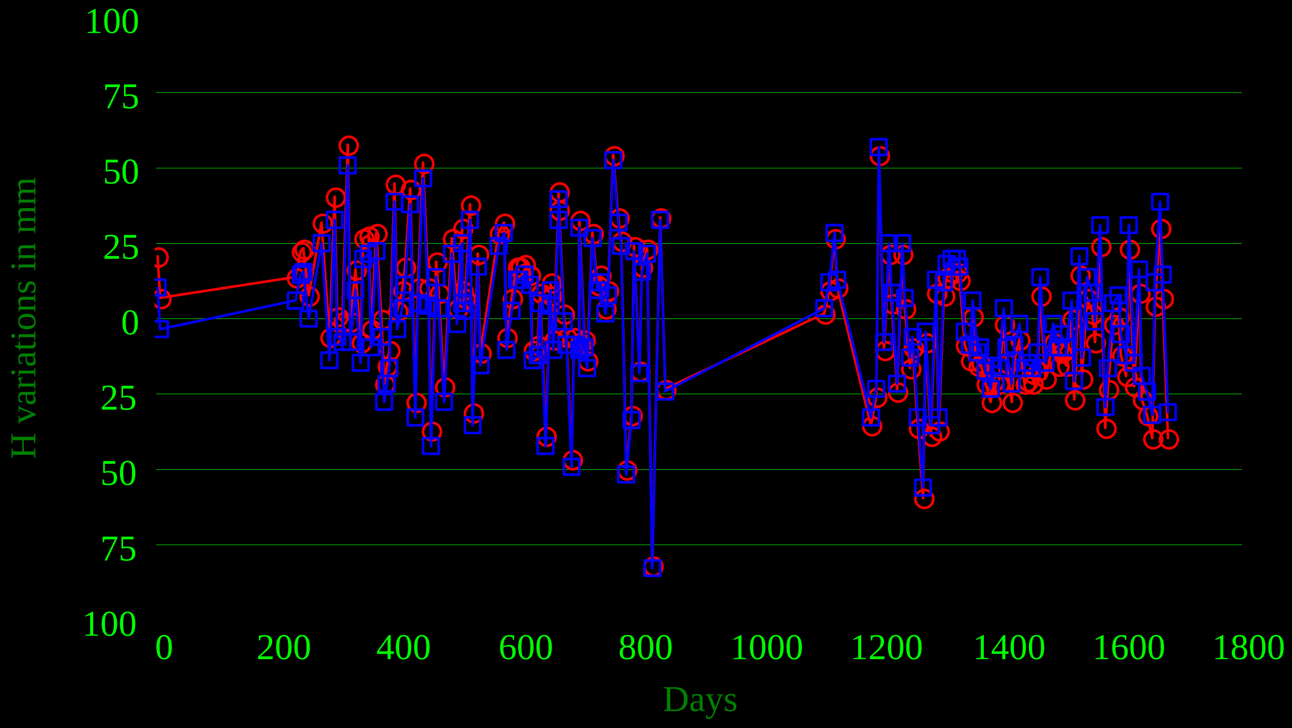
For Stations in Tetrahedron FKNW

Fortleza



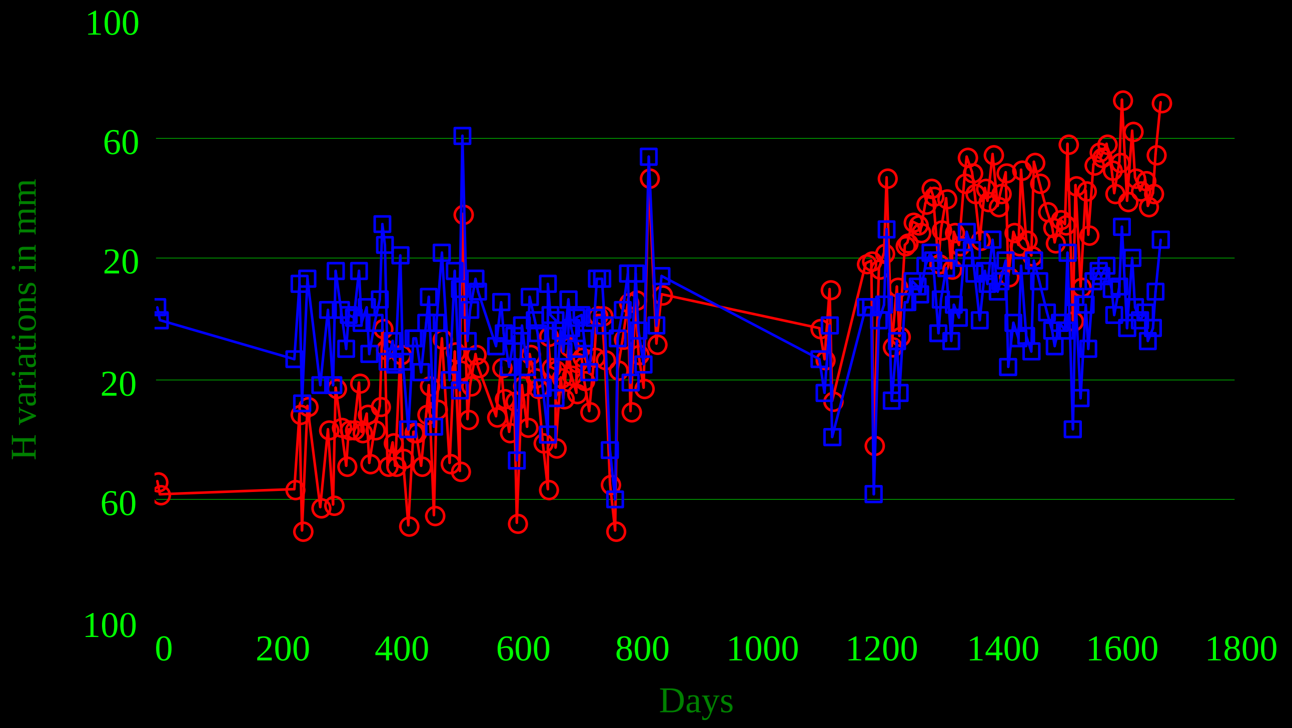
Height Variations
200
100
0
100
200
300
H variations in mm
0
200
400
600
800
1000
1200
1400
1600
1800
Days

Kokee



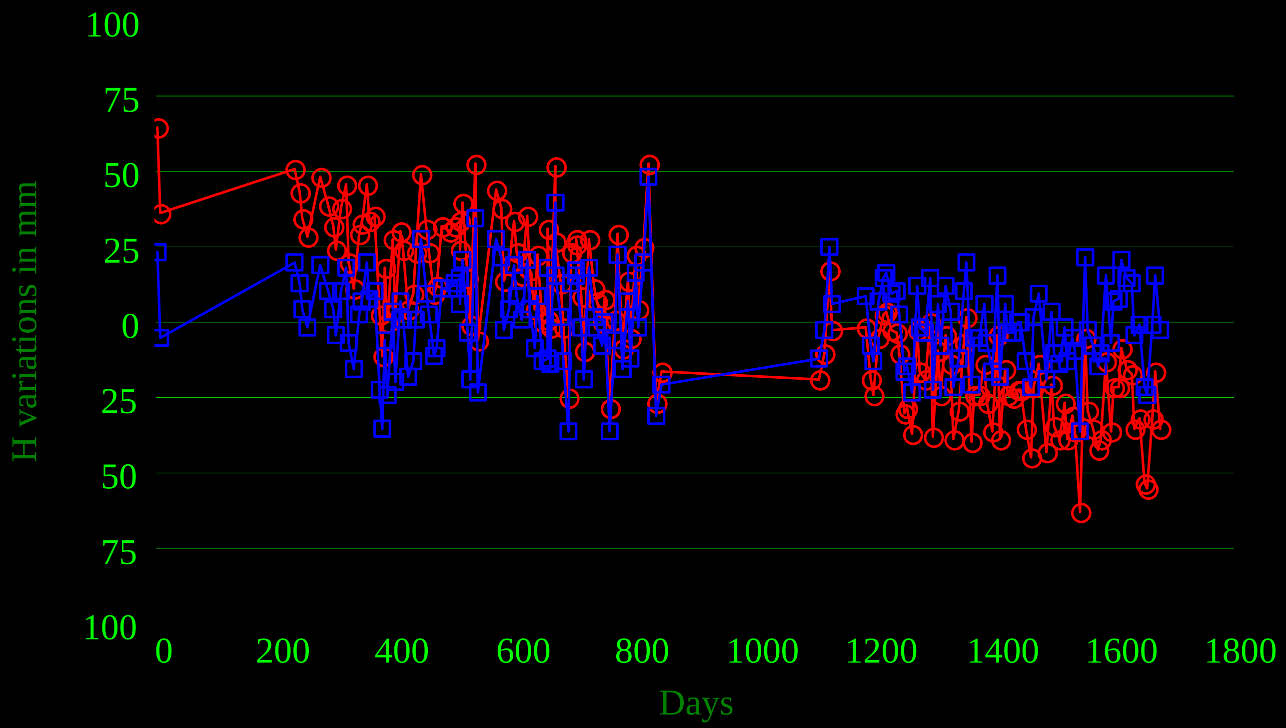
100
75
50
25
0
25
50
75
100
H variations in mm
0
200
400
600
800
1000
1200
1400
1600
1800
Days

NRAO20



100
60
20
20
60
100
H variations in mm
0
200
400
600
800
1000
1200
1400
1600
1800
Days

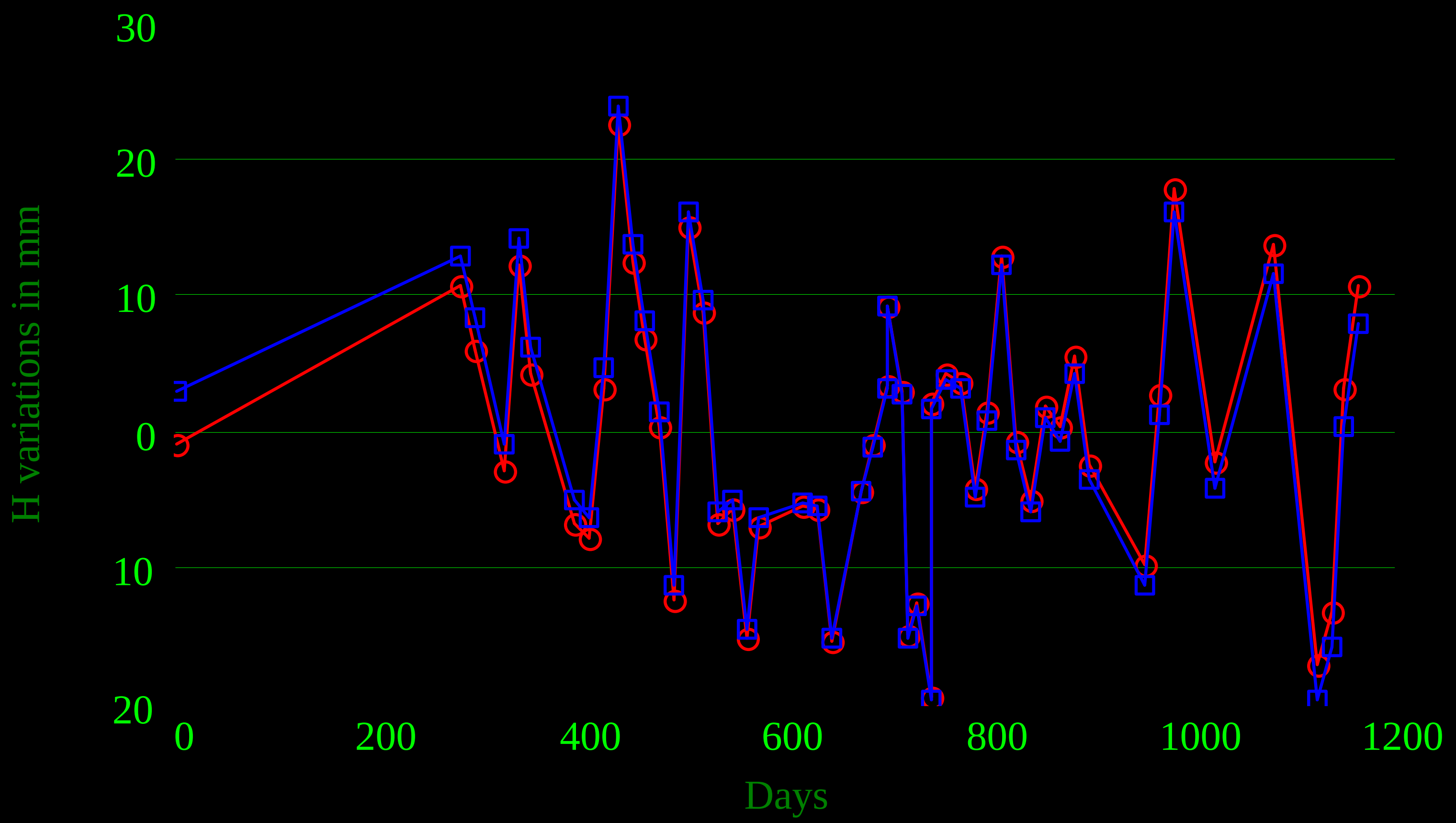
Wettzell


100
75
50
25
0
25
50
75
100
H variations in mm
0
200
400
600
800
1000
1200
1400
1600
1800
Days

Height Variations

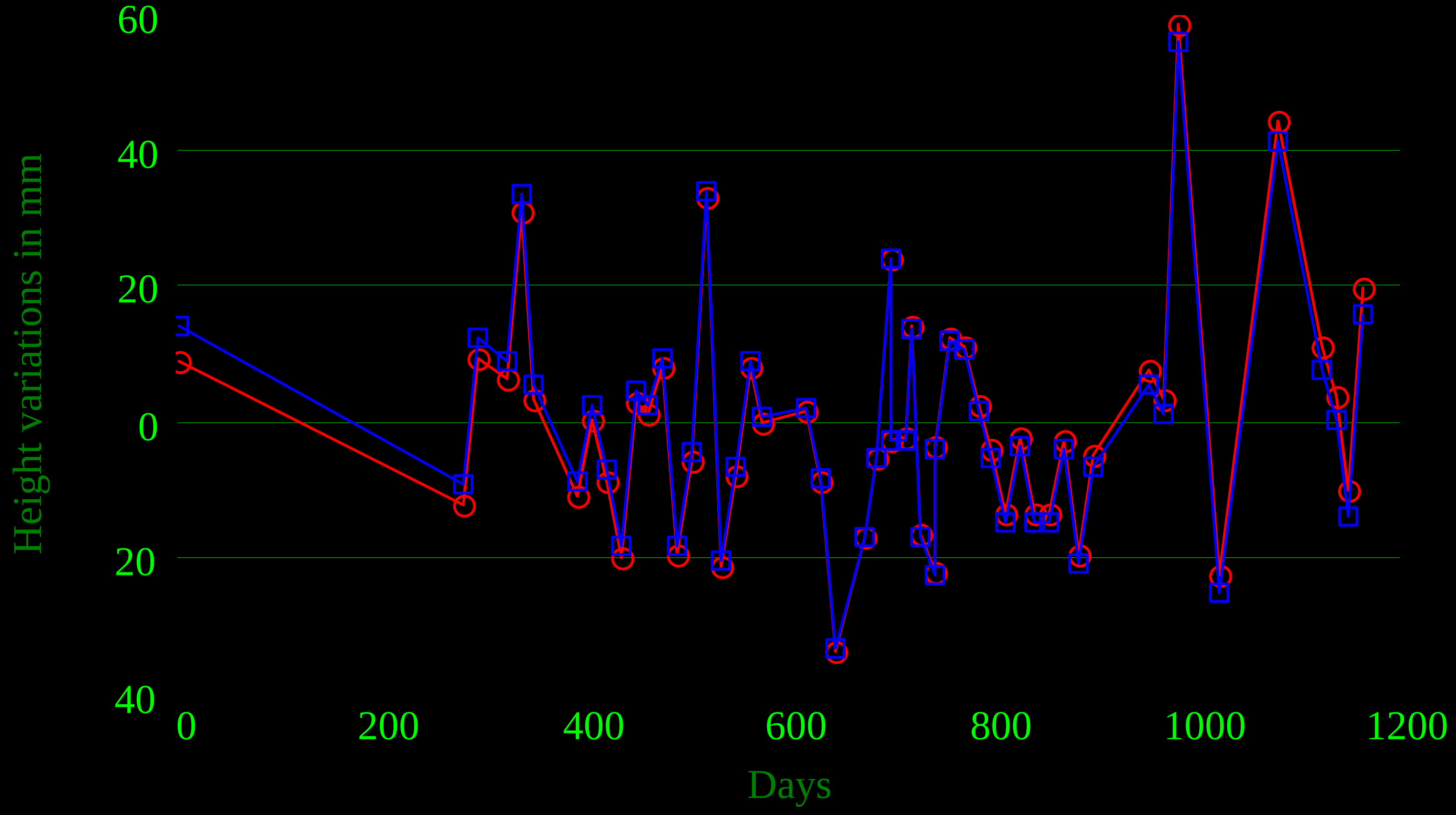
For Stations in Tetrahedron GHMW

Gilcreek

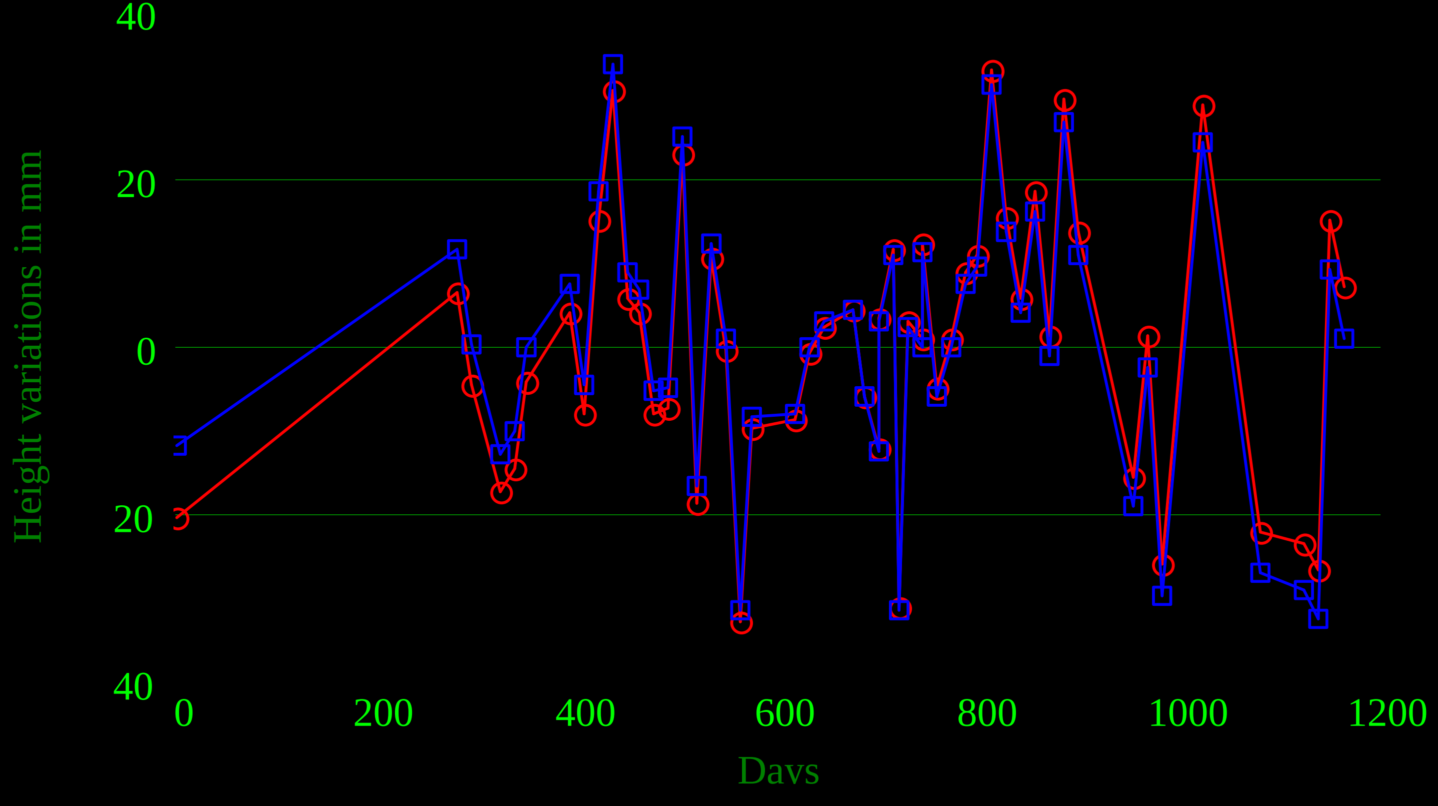


30
20
10
0
10
20
H variations in mm
0
200
400
600
800
1000
1200
Days

Hartrao

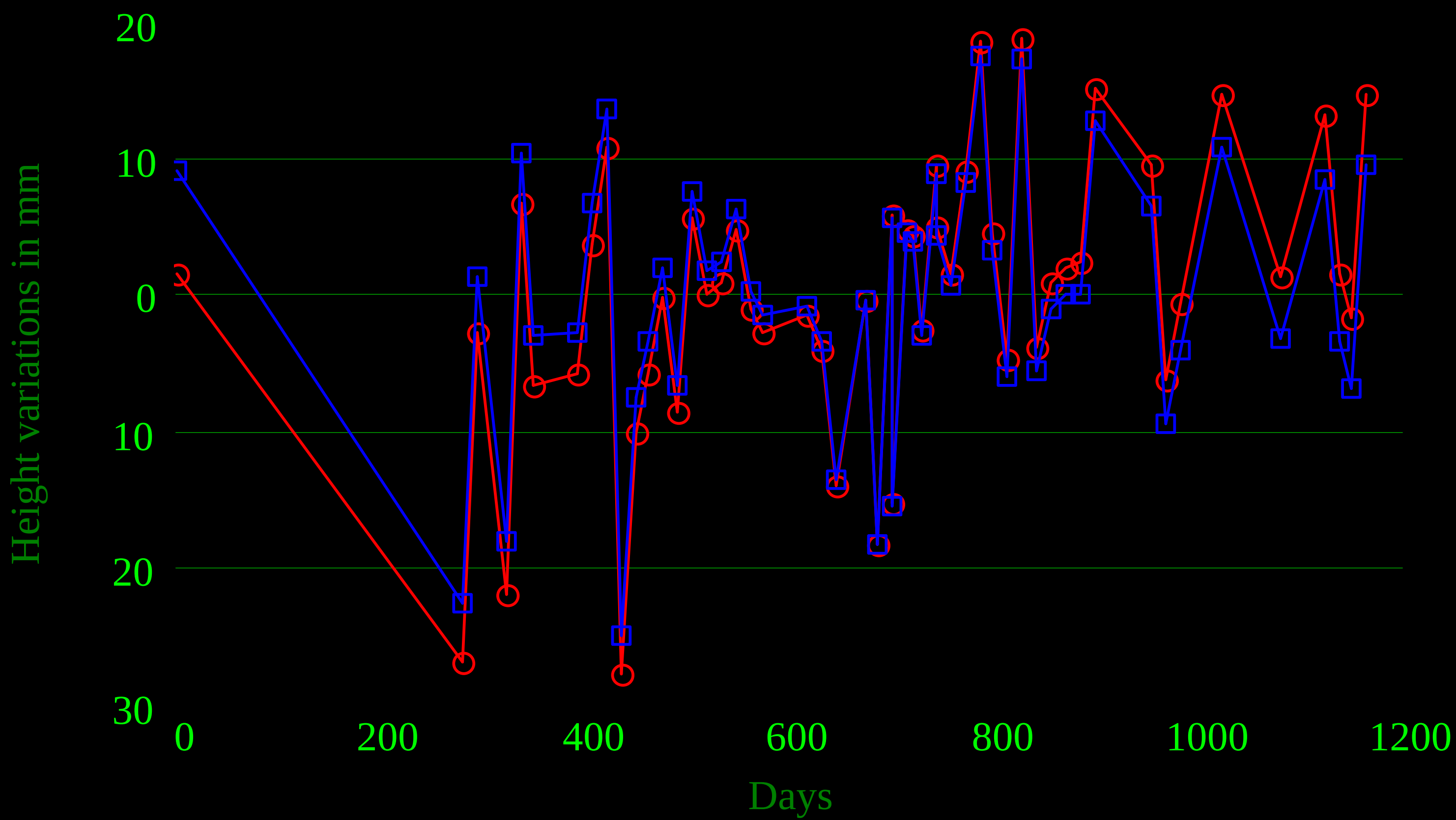

60
40
20
0
20
40
Height variations in mm
0
200
400
600
800
1000
1200
Days

Matera



Height variations in mm
40
20
0
20
40
0
200
400
600
800
1000
1200
Days

Westford

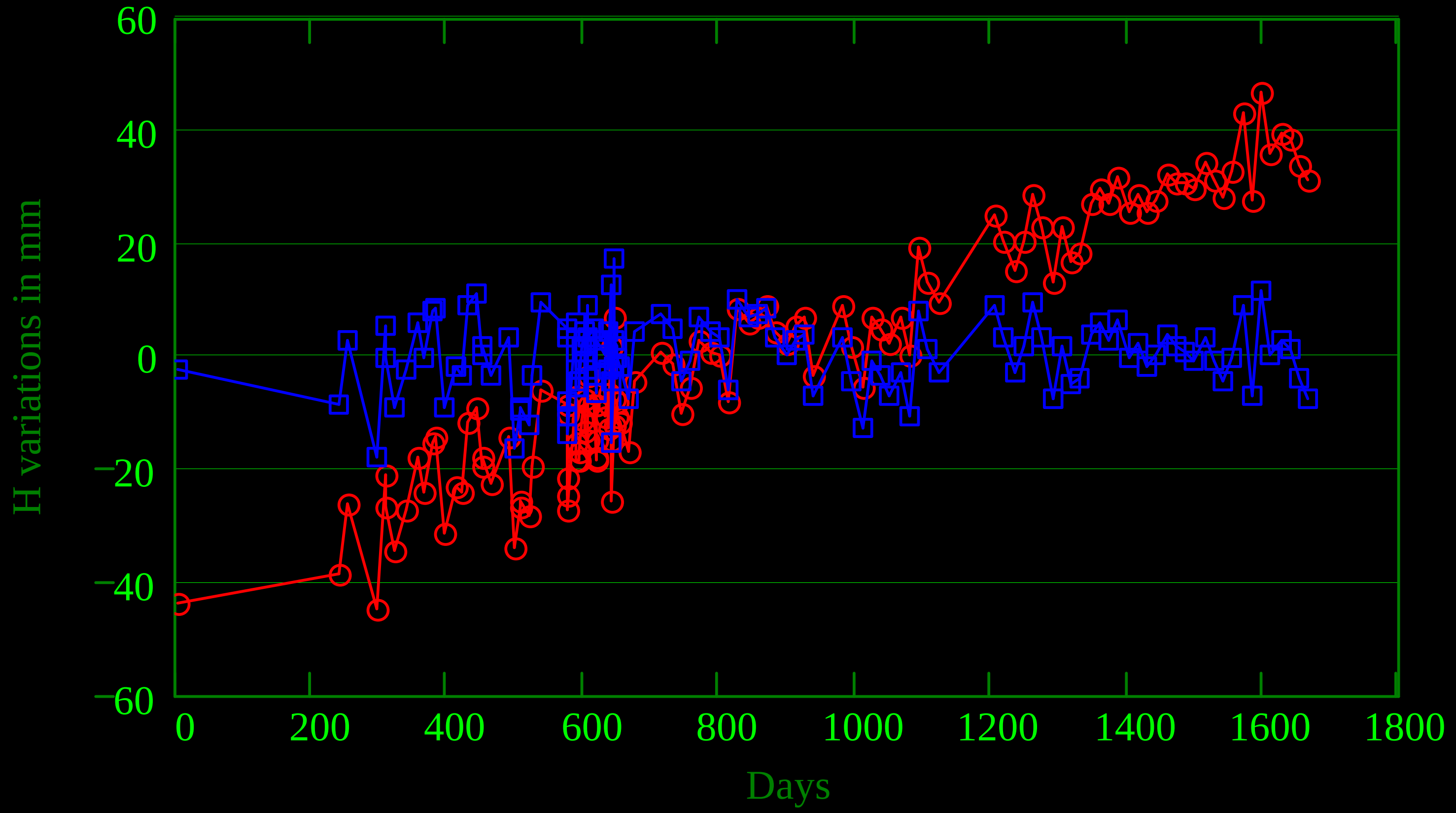


20
10
0
10
20
30
Height variations in mm
0
200
400
600
800
1000
1200
Days

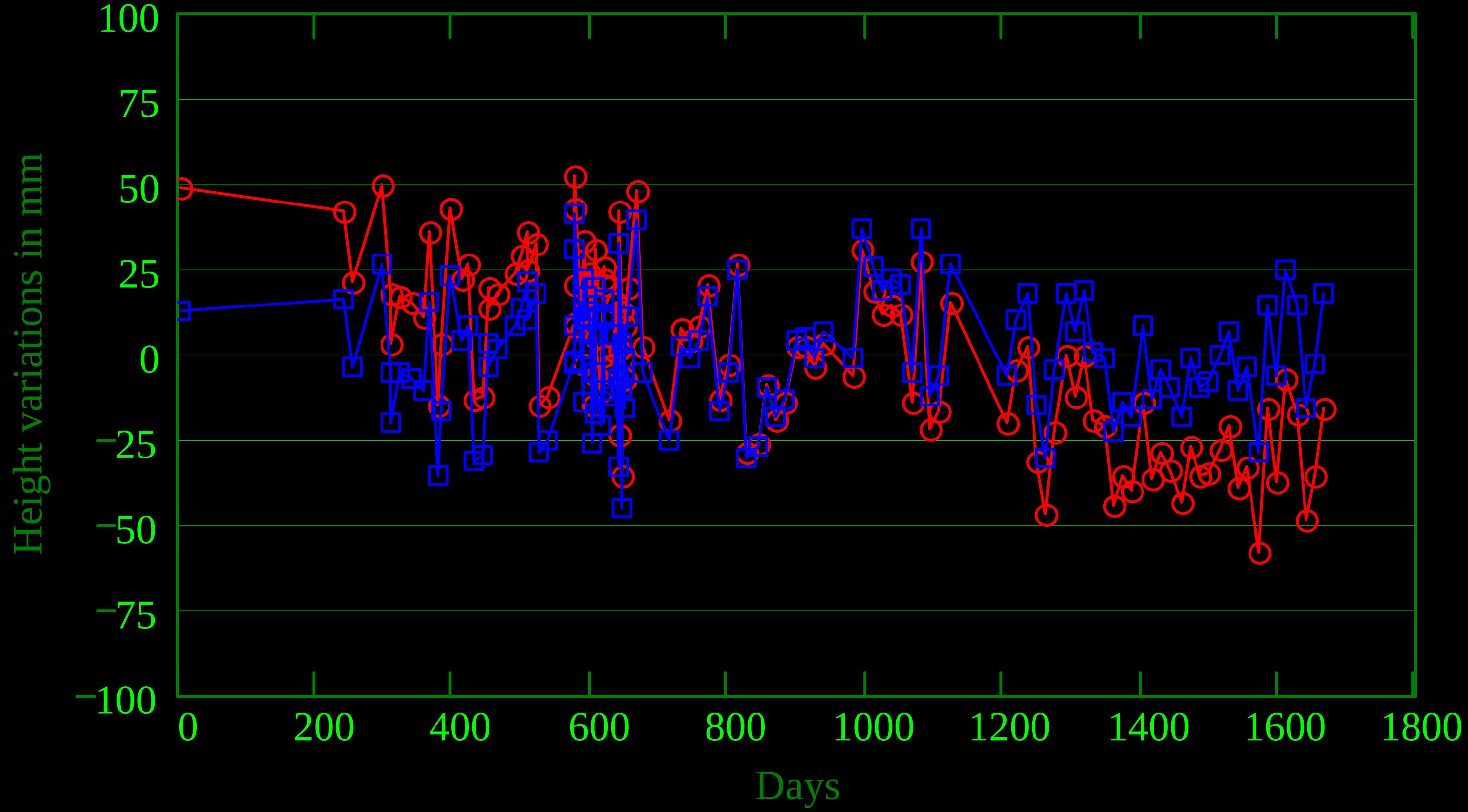
Height Variations

For Stations in Tetrahedron GKNW

Gilcreek

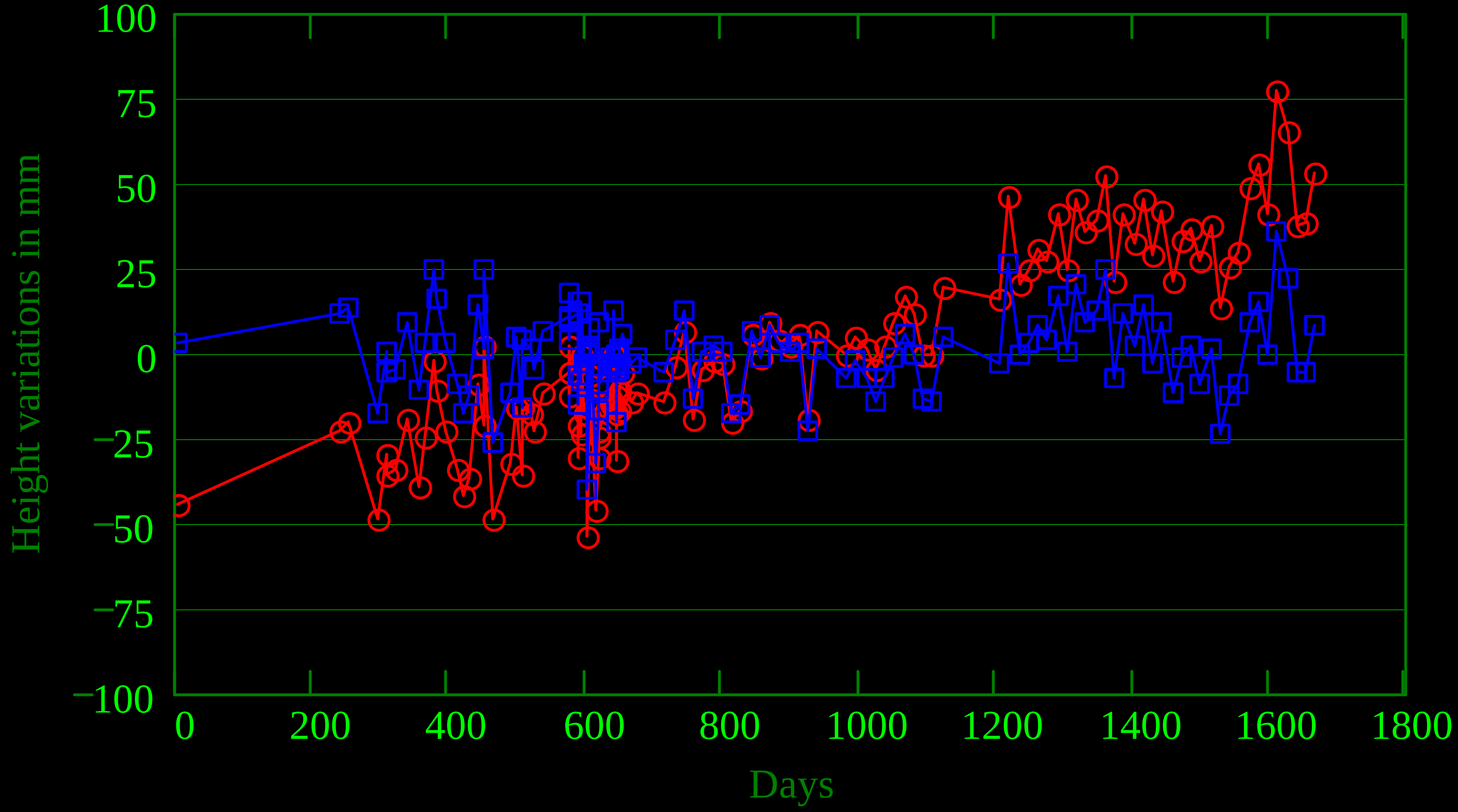

60
40
20
0
−20
−40
−60
0
200
400
600
800
1000
1200
1400
1600
1800
H variations in mm
Days

Kokee



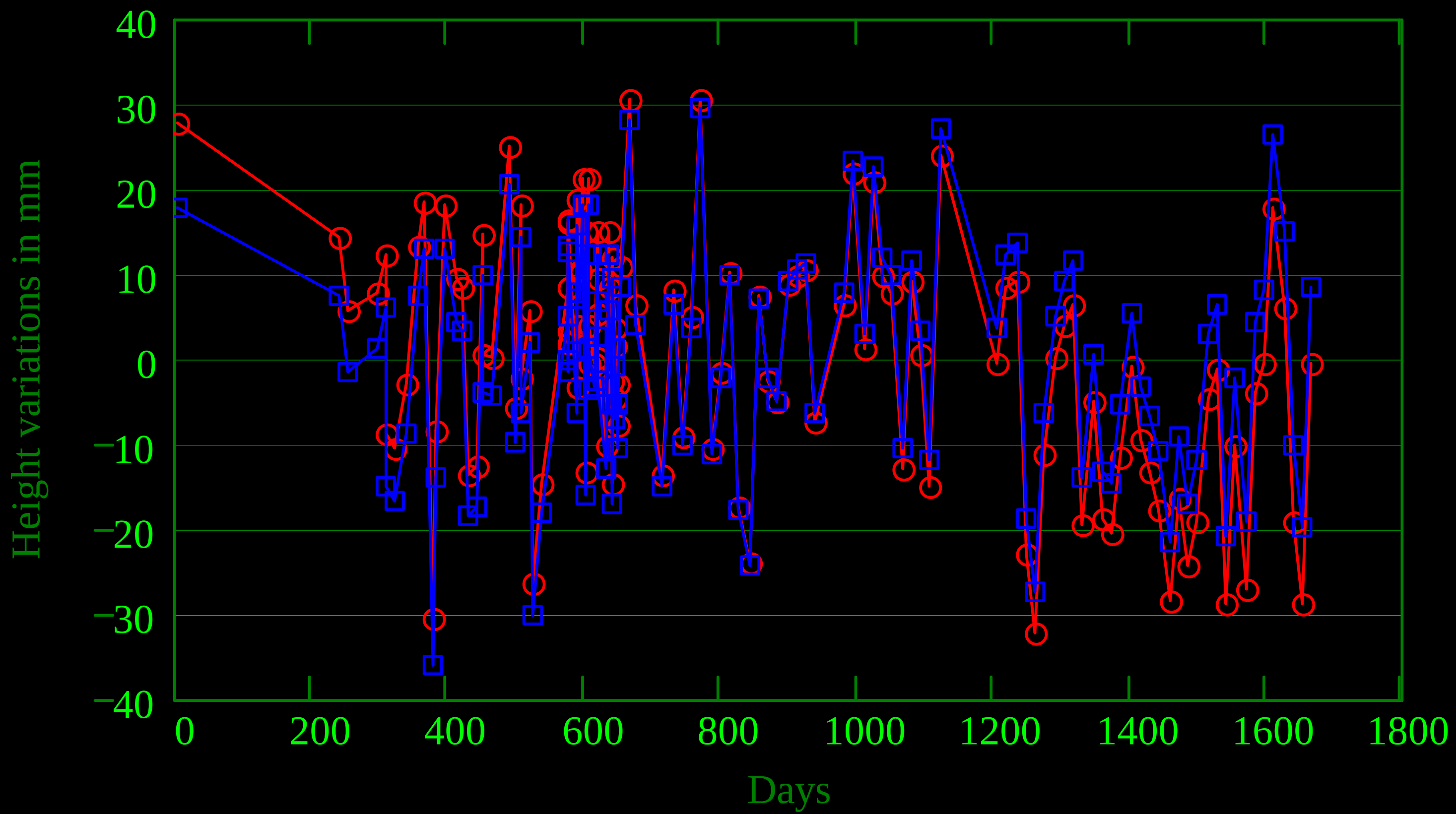
100
75
50
25
0
−25
−50
−75
−100
0
200
400
600
800
1000
1200
1400
1600
1800
Height variations in mm
Days

NRAO20



100
75
50
25
0
−25
−50
−75
−100
Height variations in mm
0
200
400
600
800
1000
1200
1400
1600
1800
Days

Wettzell



40
30
20
10
0
−10
−20
−30
−40
0
200
400
600
800
1000
1200
1400
1600
1800
Height variations in mm
Days

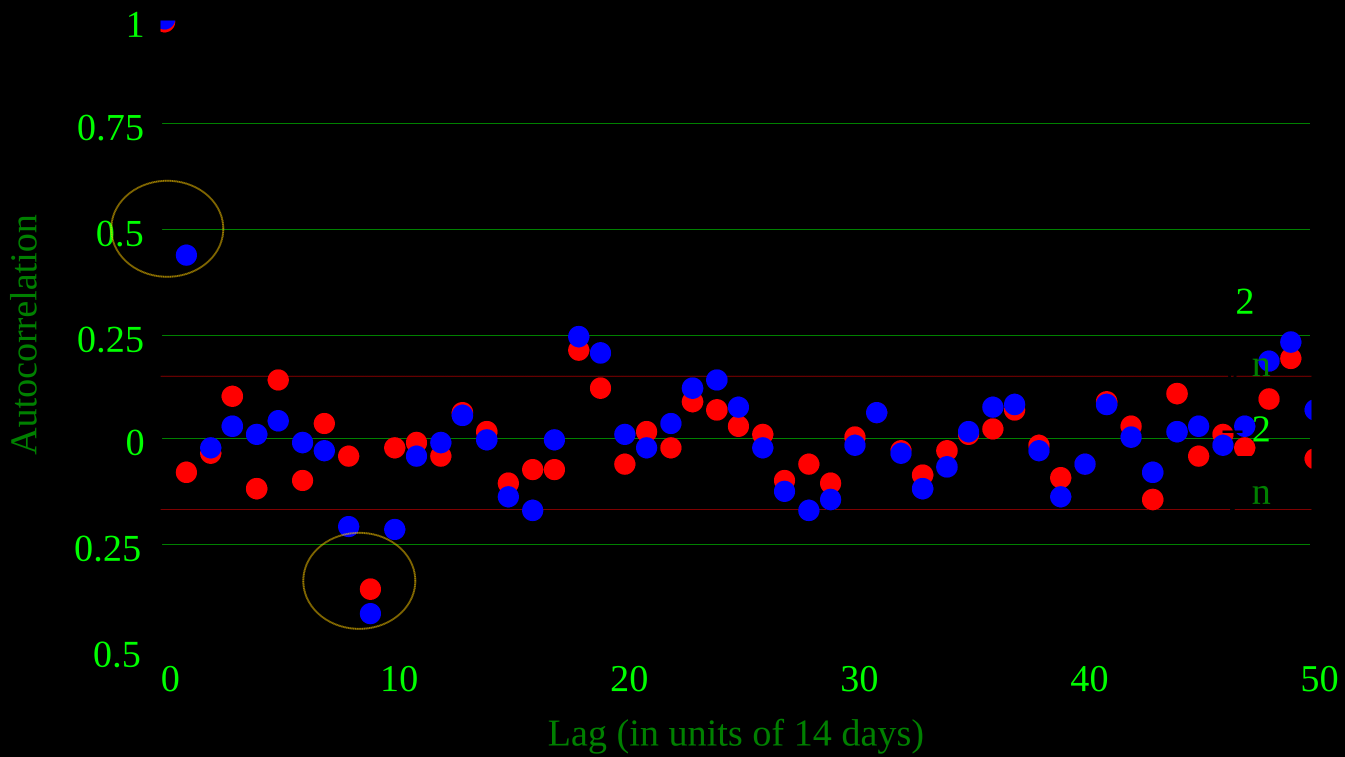
Turning point test for randomness

Network	Station	Counted	Expected Count	Standard Deviation
FKNW	Fortleza	99	106.0	5.3
	Kokee	106		
	NRAO20	107		
	Wettzell	100		
GHMW	Gilcreek	28	33.3	3.0
	Hartrao	33		
	Matera	29		
	Westford	33		
GKNW	Gilcreek	78	81.3	4.6
	Kokee	84		
	NRAO20	84		
	Wettzell	73		

Autocorrelations

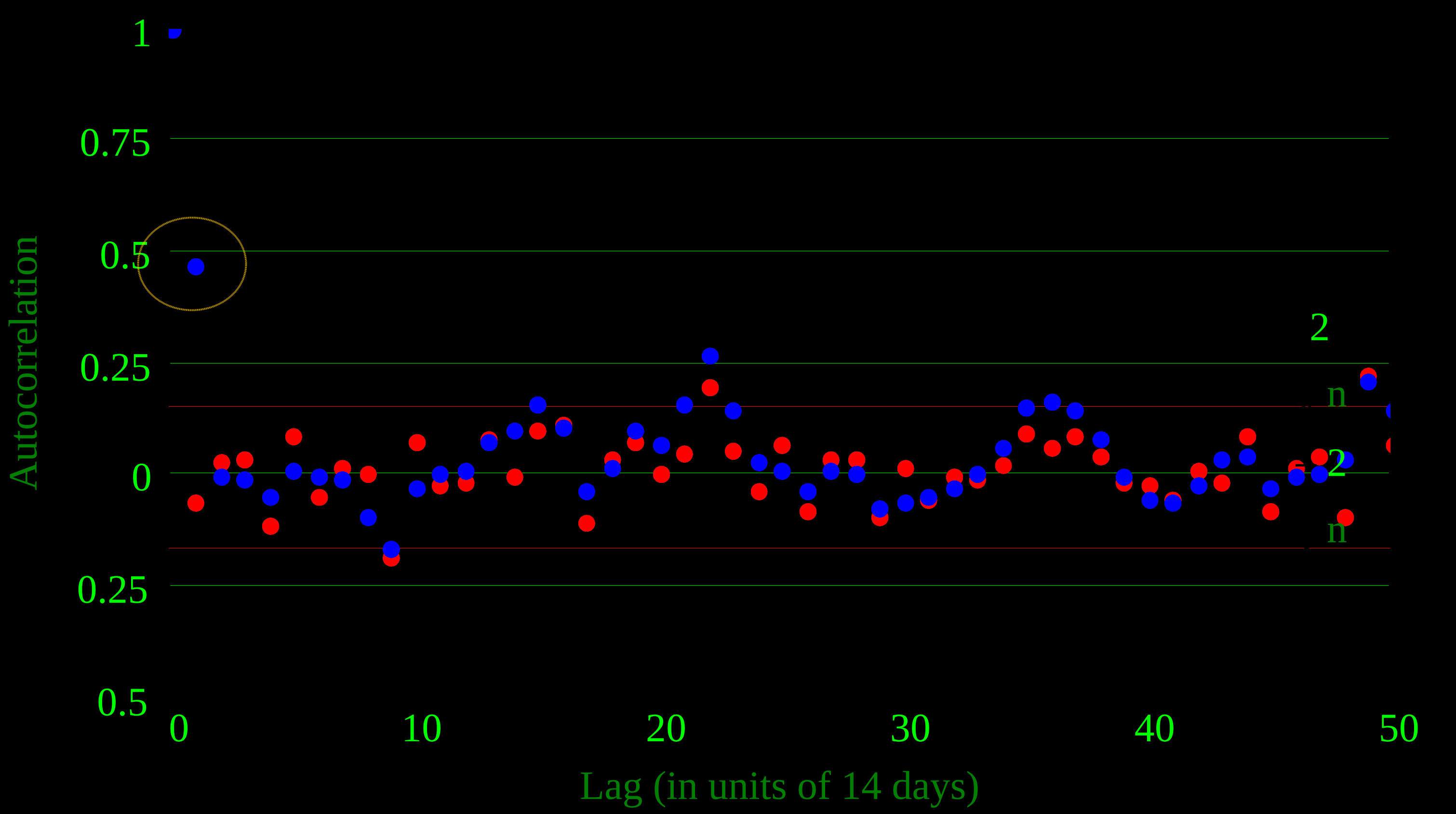
Cumulative and interval series
for tetrahedron: FKNW

Fortleza



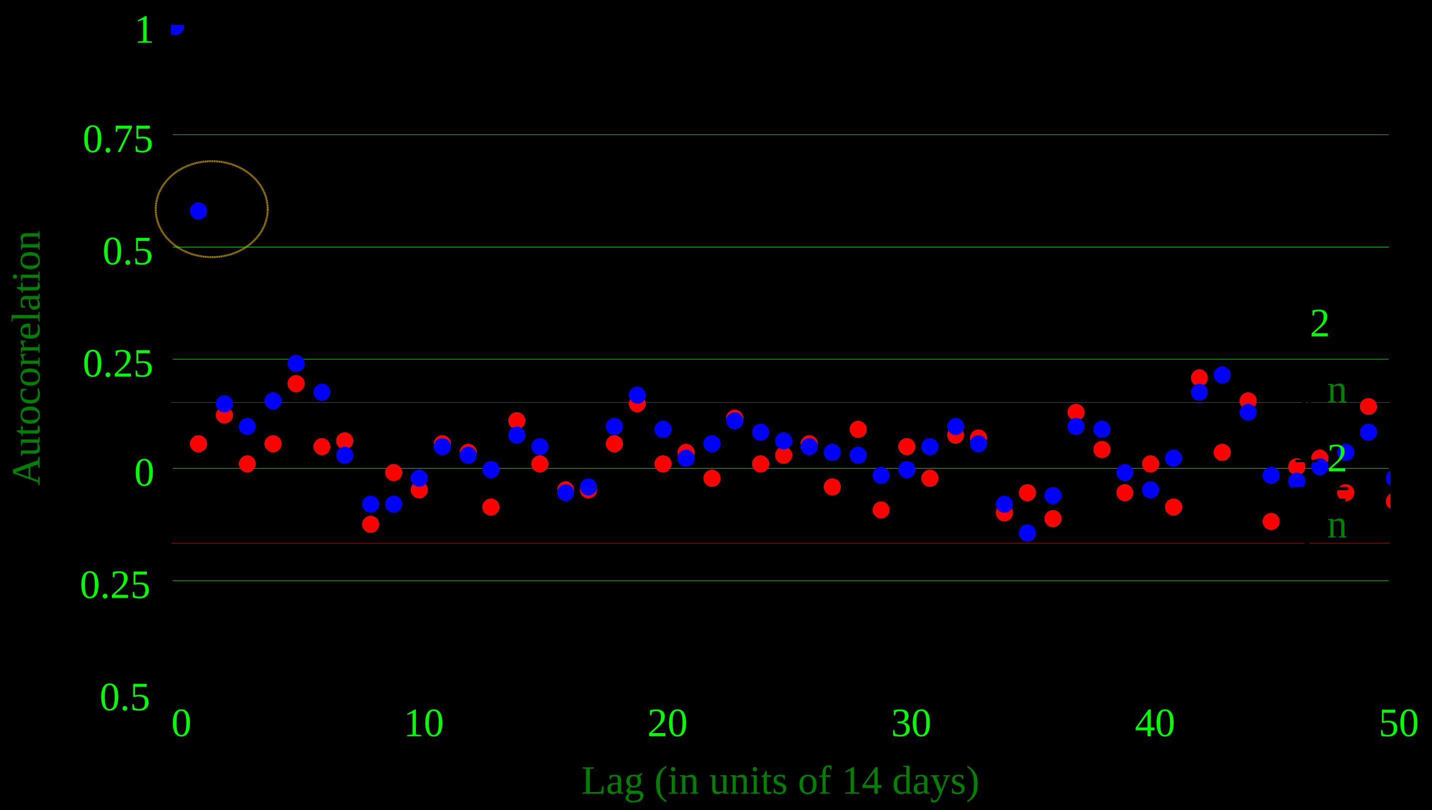
1
0.75
0.5
0.25
0
0.25
0.5
Autocorrelation
2
n
2
n
0
10
20
30
40
50
Lag (in units of 14 days)

Kokee



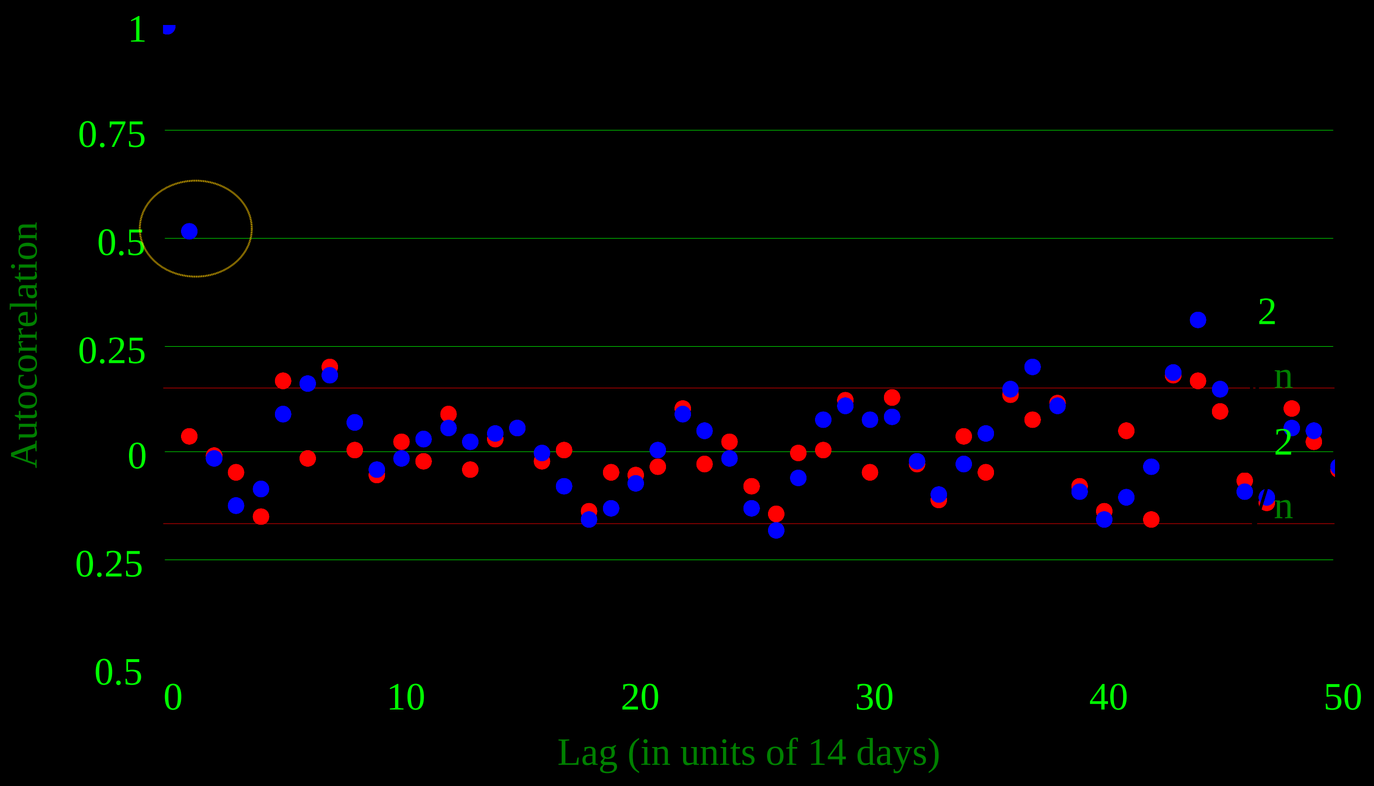
1
0.75
0.5
0.25
0
0.25
0.5
Autocorrelation
0
10
20
30
40
50
Lag (in units of 14 days)
2
n
2
n

NRAO20



1
0.75
0.5
0.25
0
0.25
0.5
Autocorrelation
0
10
20
30
40
50
Lag (in units of 14 days)
2
n
2
n

Wettzell

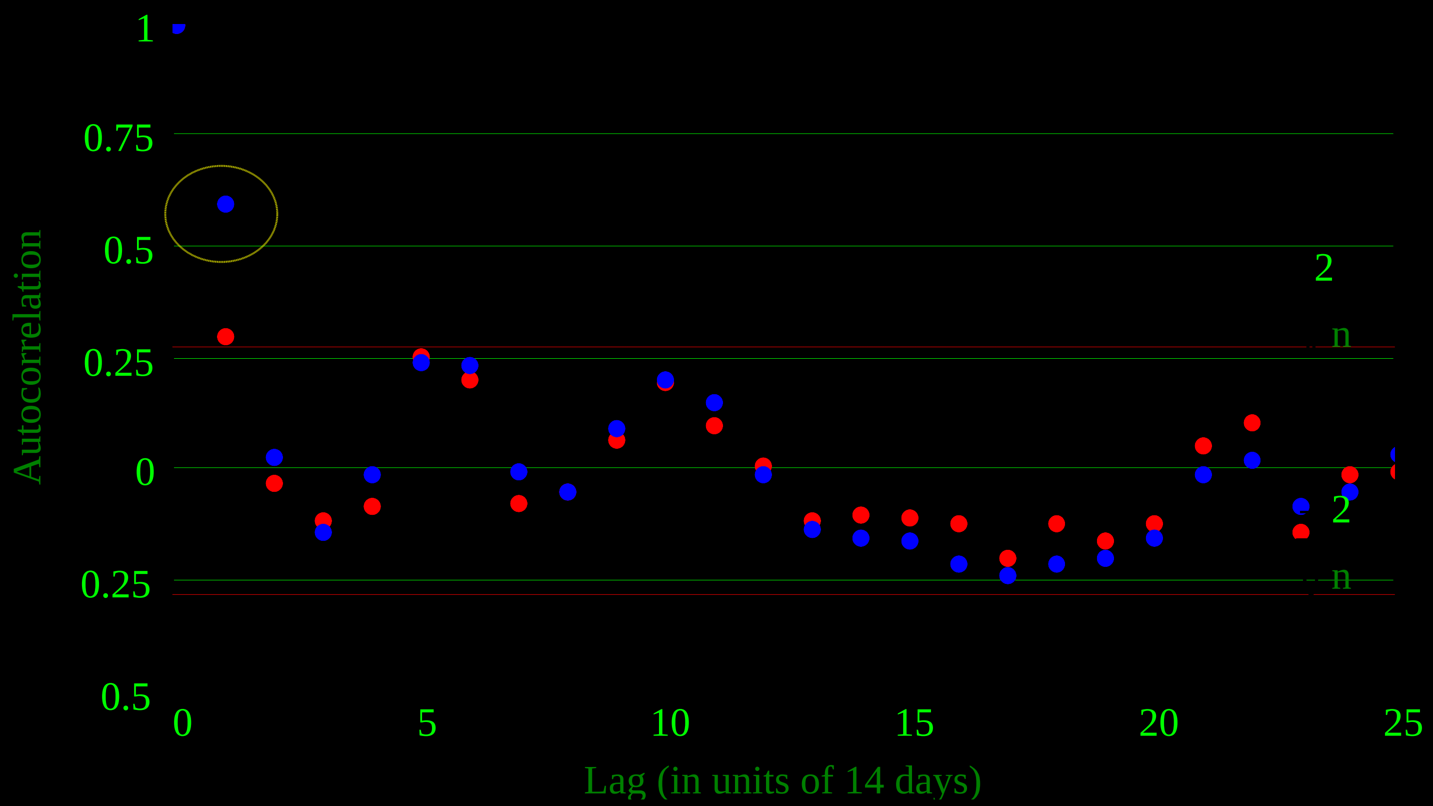


1
0.75
0.5
0.25
0
0.25
0.5
2
n
2
n
0
10
20
30
40
50
Autocorrelation
Lag (in units of 14 days)

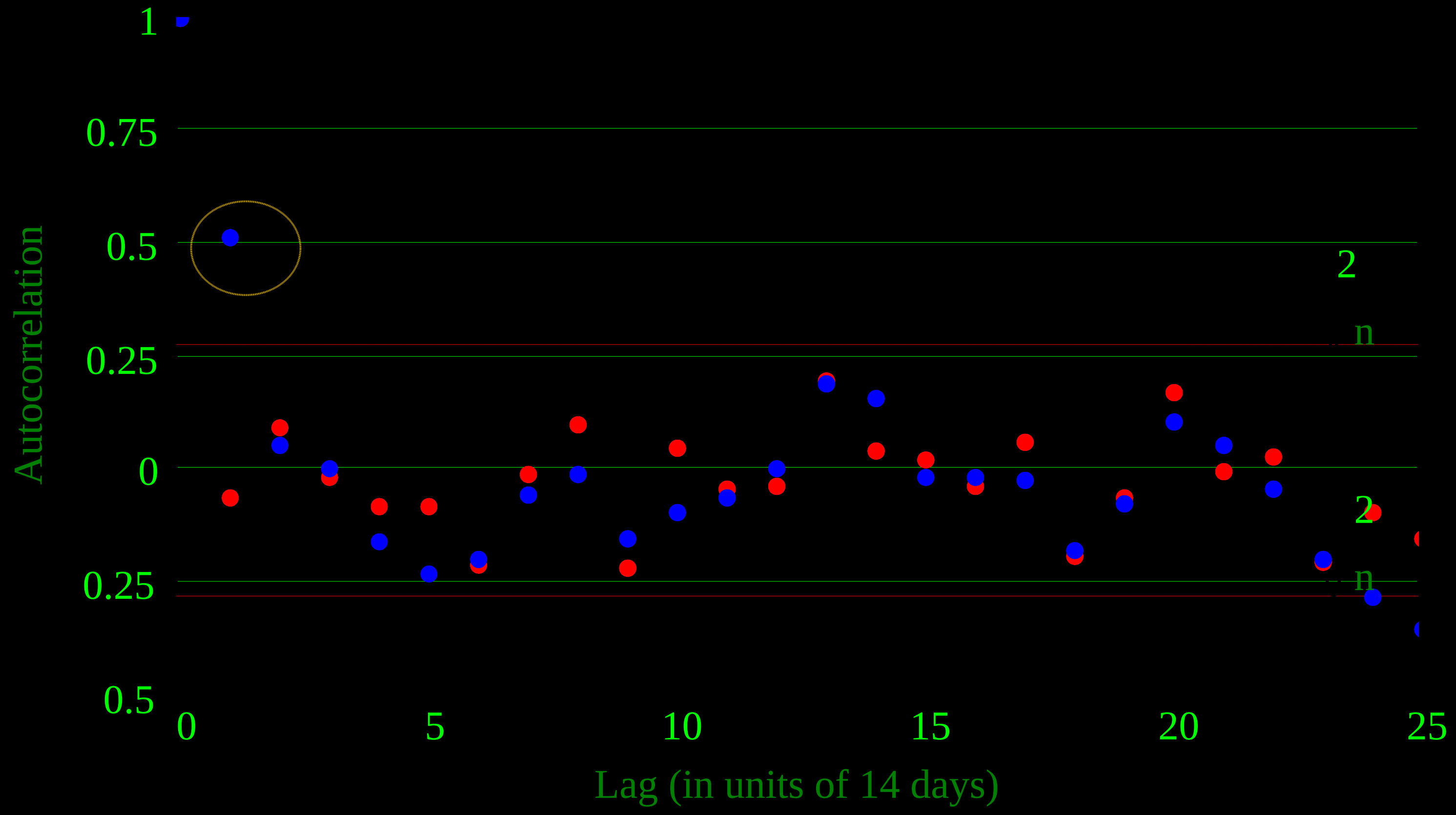
Autocorrelations

Cumulative and interval series
for tetrahedron: GHMW

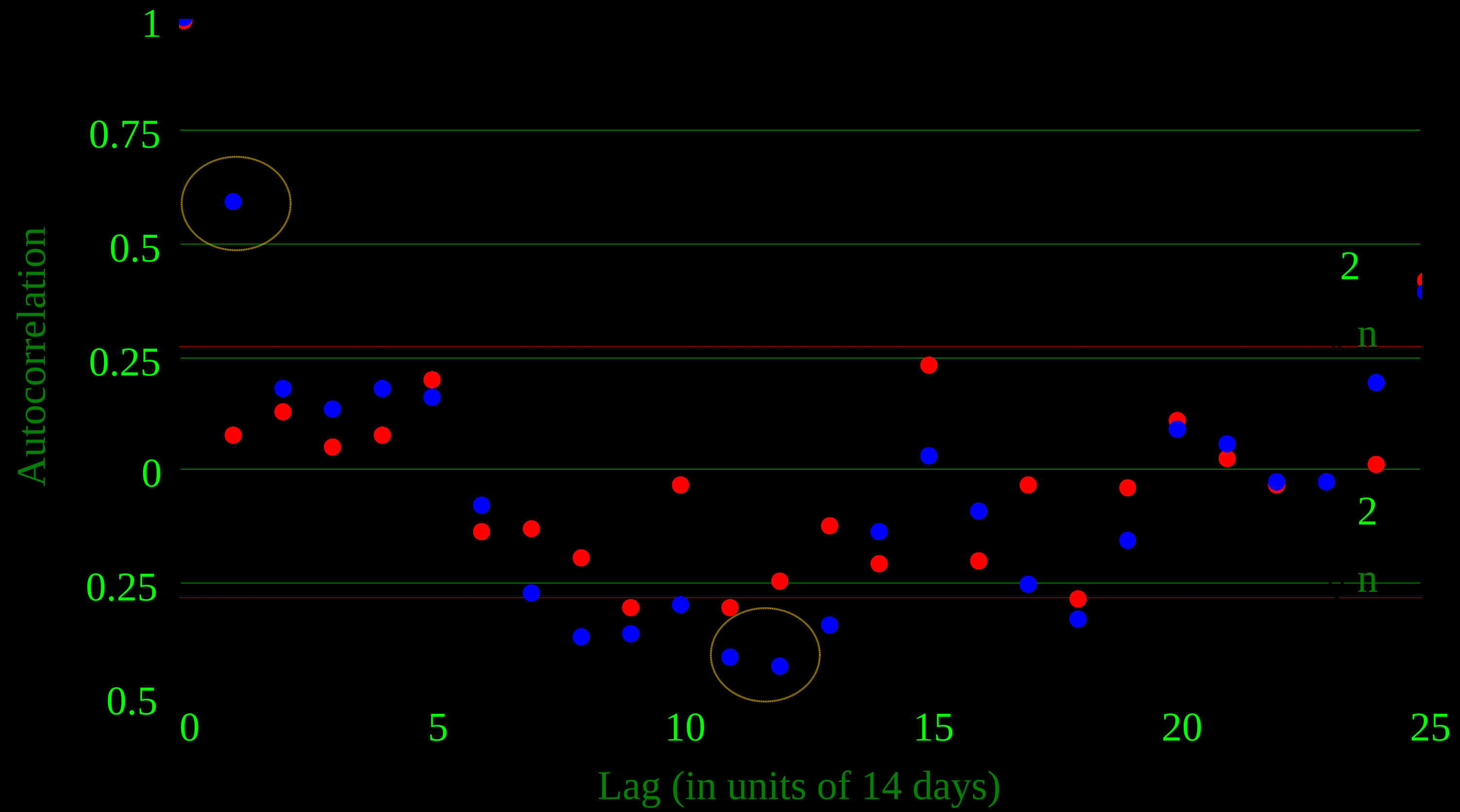
Gilcreek


1
0.75
0.5
0.25
0
0.25
0.5
Autocorrelation
2
n
2
n
0
5
10
15
20
25
Lag (in units of 14 days)

Hartrao

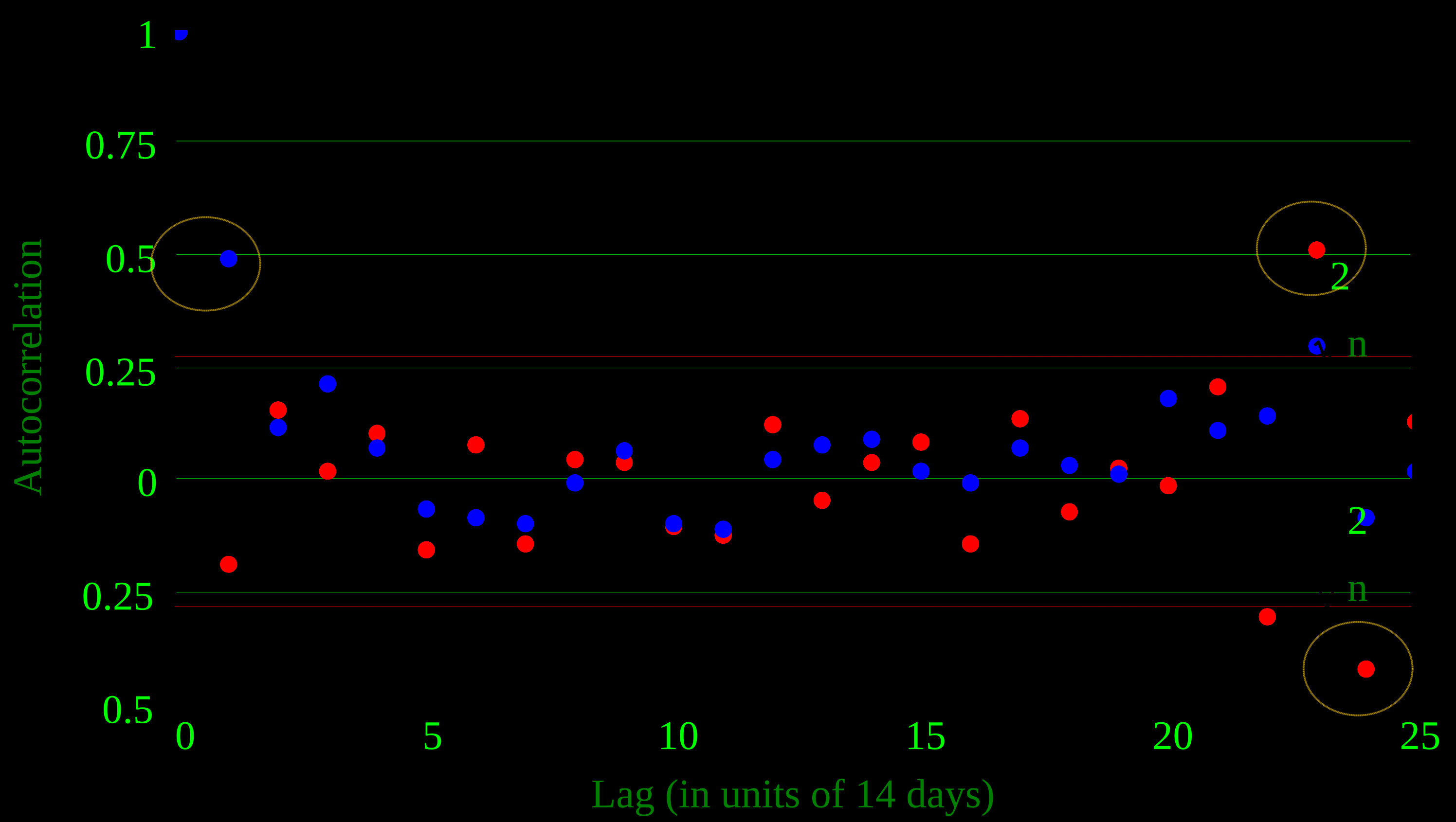

1
0.75
0.5
0.25
0
0.25
0.5
Autocorrelation
2
n
2
n
0
5
10
15
20
25
Lag (in units of 14 days)

Matera



1
0.75
0.5
0.25
0
0.25
0.5
Autocorrelation
0
5
10
15
20
25
Lag (in units of 14 days)
2
n
2
n

Westford

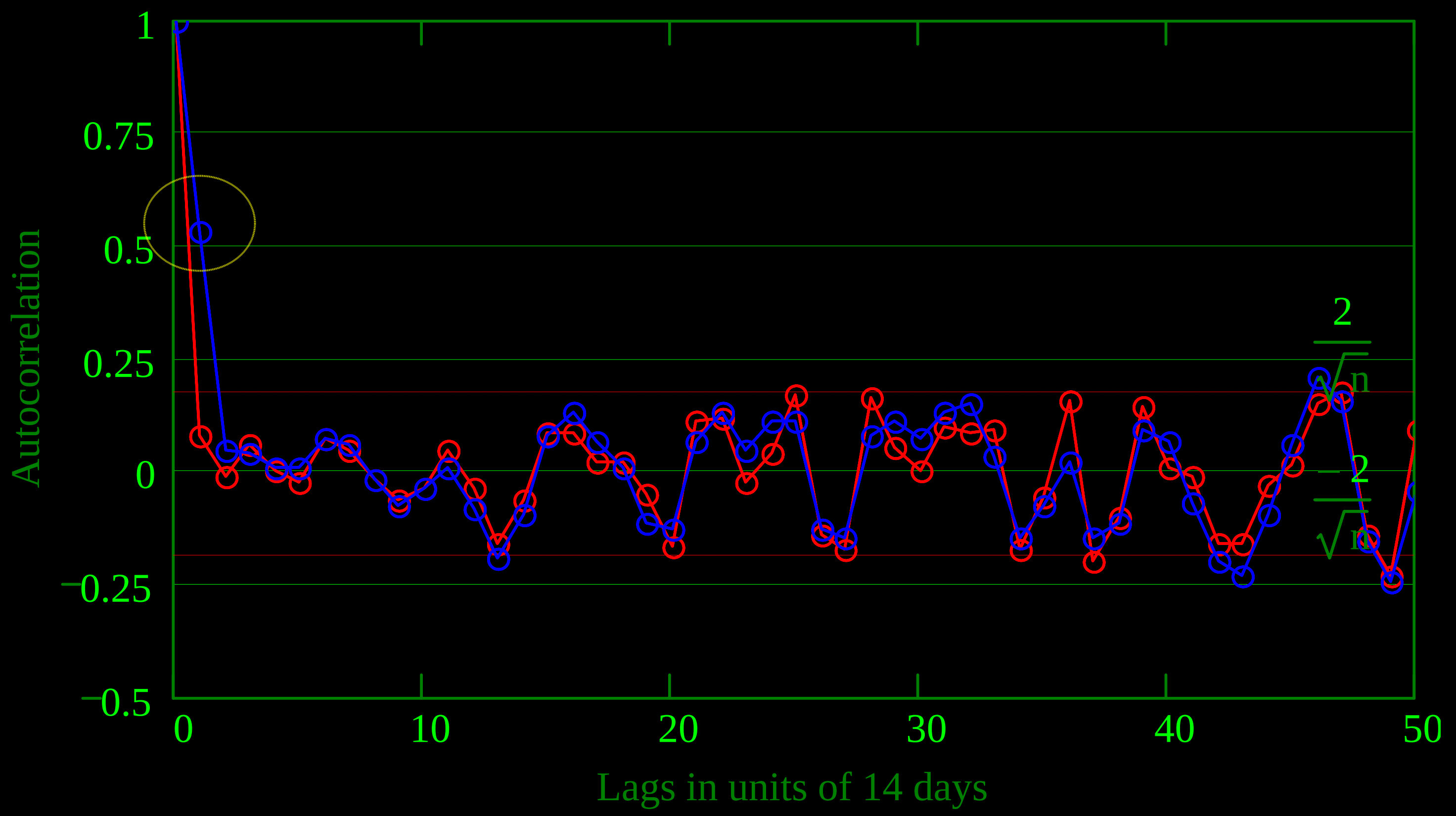


1
0.75
0.5
0.25
0
0.25
0.5
Autocorrelation
0
5
10
15
20
25
Lag (in units of 14 days)
2
n
2
n

Autocorrelations

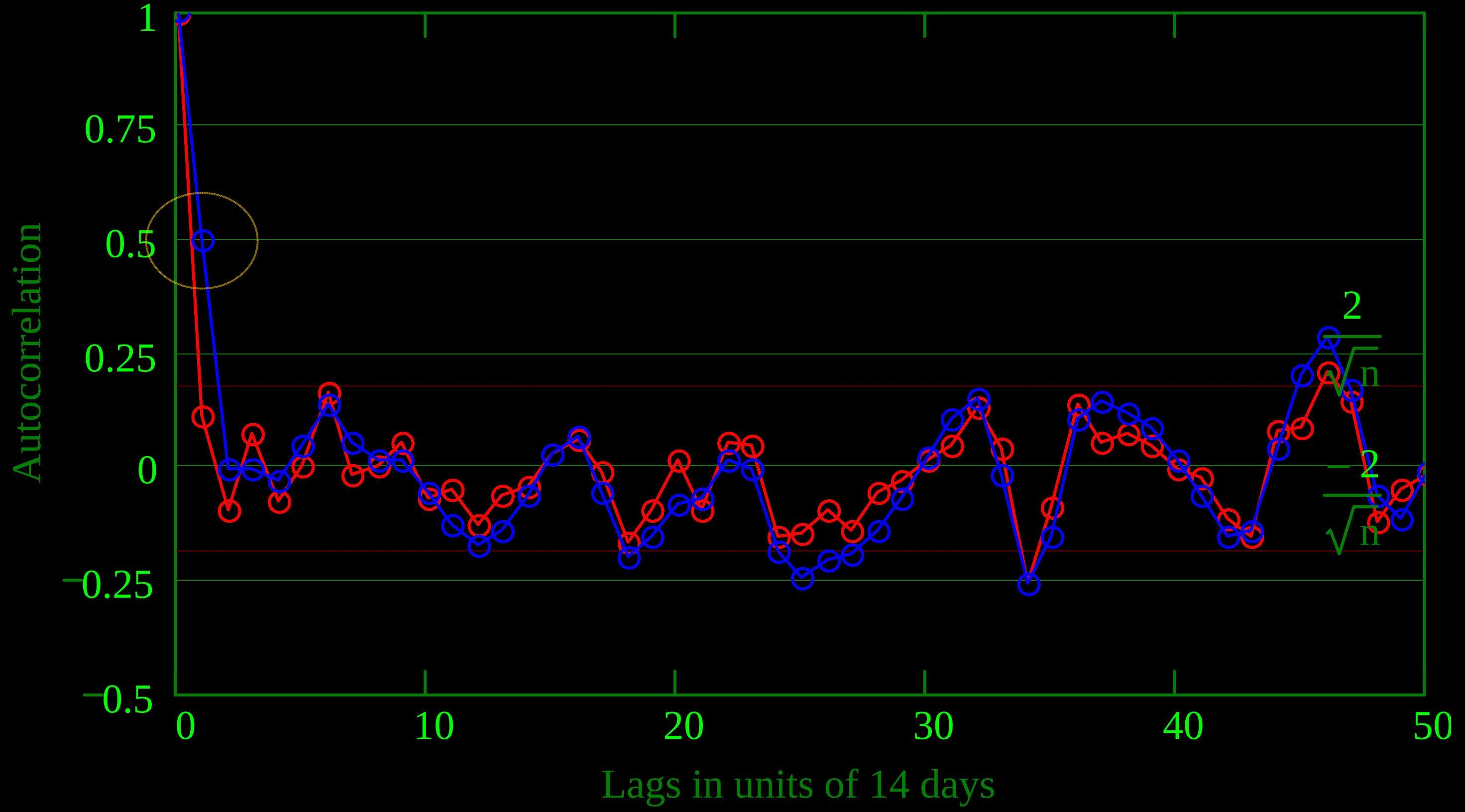
Cumulative and interval series
for tetrahedron: GKNW

Gilcreek



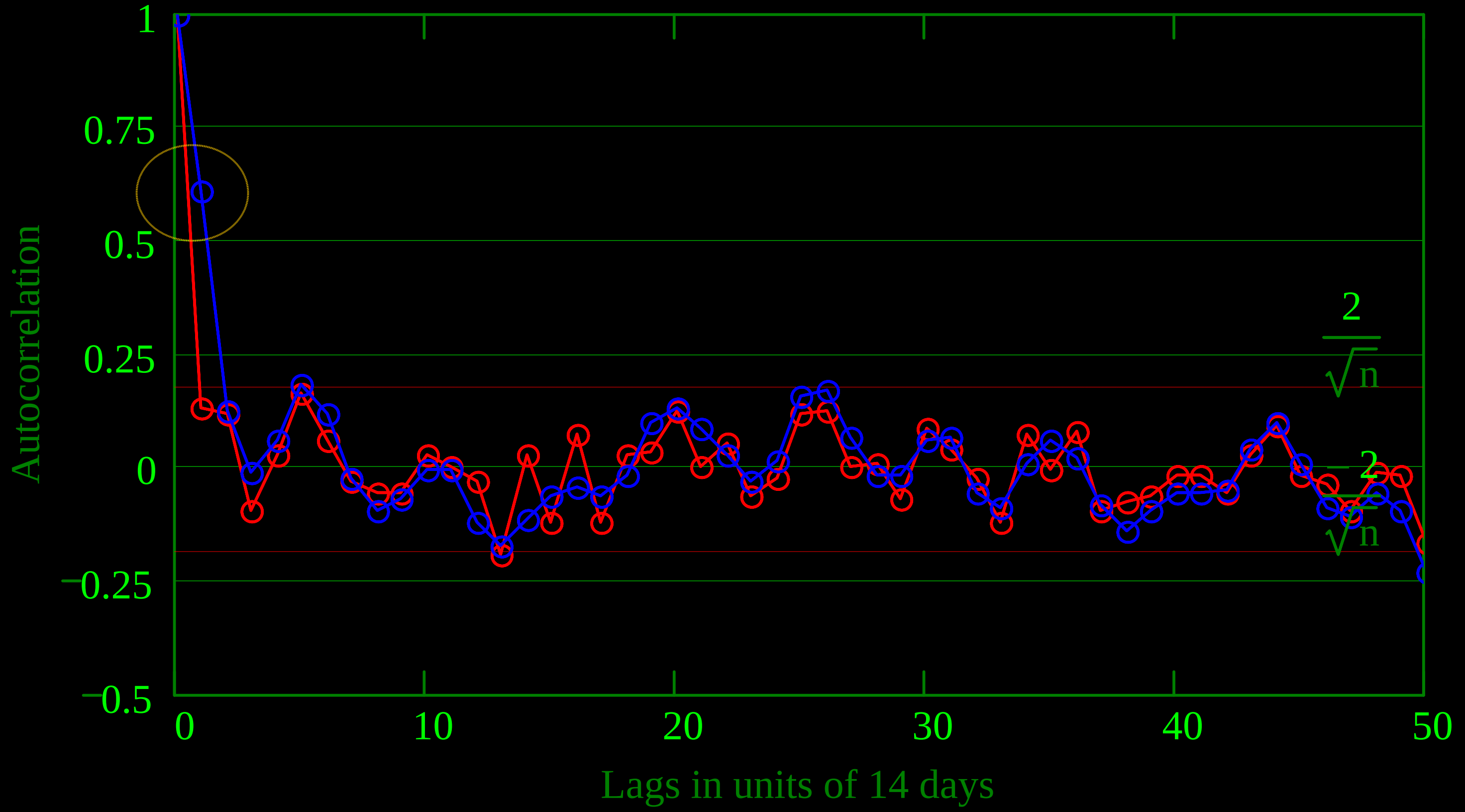
1
0.75
0.5
0.25
0
−0.25
−0.5
Autocorrelation
0
10
20
30
40
50
Lags in units of 14 days
2
√n
− 2
√n

Kokee



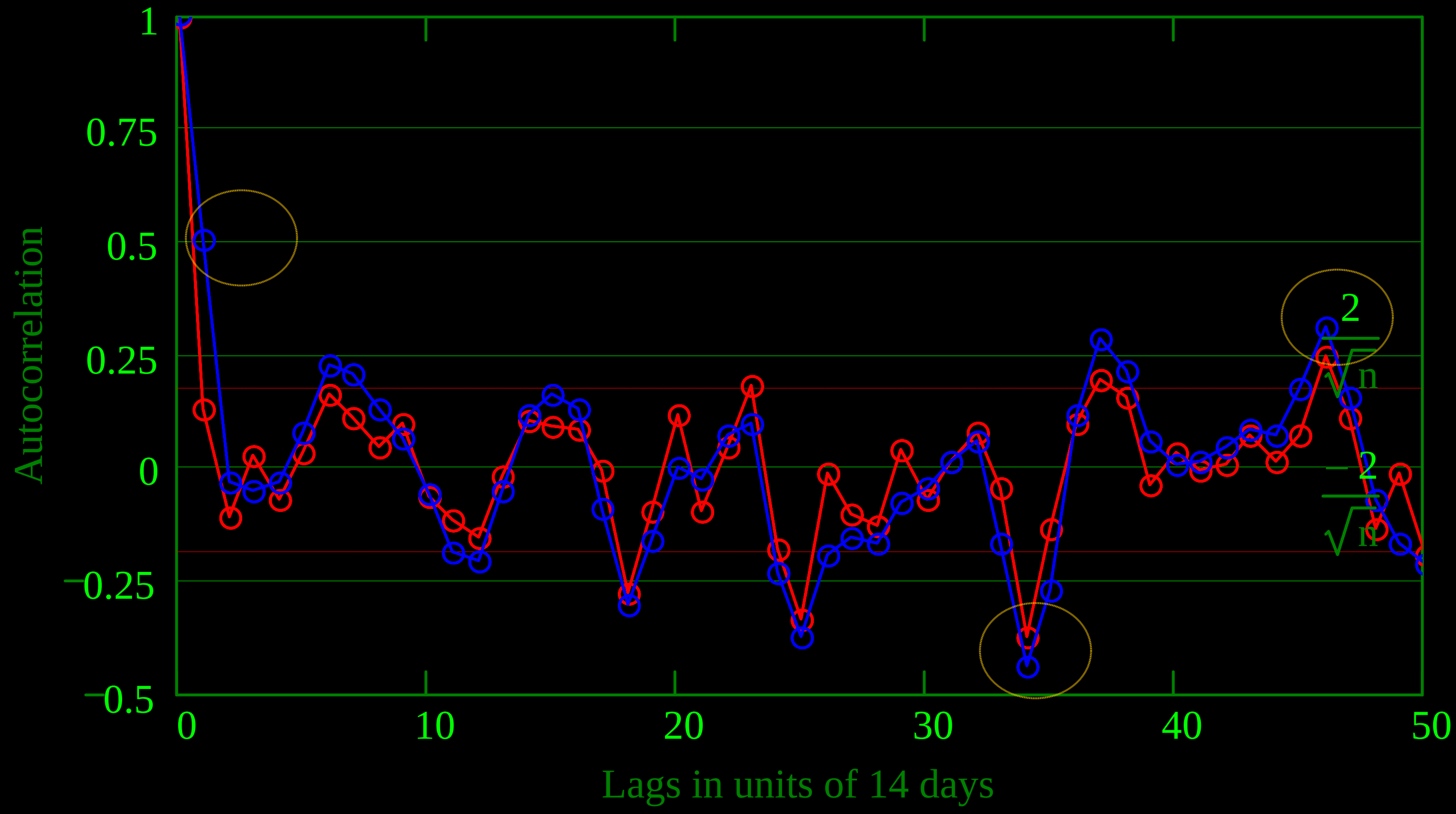
1
0.75
0.5
0.25
0
−0.25
−0.5
0
10
20
30
40
50
Autocorrelation
Lags in units of 14 days
$\frac{2}{\sqrt{n}}$
$-\frac{2}{\sqrt{n}}$

NRAO20



1
0.75
0.5
0.25
0
−0.25
−0.5
Autocorrelation
0
10
20
30
40
50
Lags in units of 14 days
2
n
−2
n

Wettzell



1
0.75
0.5
0.25
0
−0.25
−0.5
Autocorrelation
0
10
20
30
40
50
Lags in units of 14 days
2
√n
−2
√n

Summary statistics

Tetrahedron	Station	N	s (*) (mm)	$\bar{s}$ (**) (mm)
FKNW	Fortleza	161	34.1	2.8
	Kokee	161	22.6	1.8
	NRAO20	161	18.1	1.4
	Wettzell	161	15.4	1.2
GHMW	Gilcreek	52	9.8	1.4
	Hartrao	52	17.1	2.4
	Matera	52	16.0	2.2
	Westford	52	9.4	1.3
GKNW	Gilcreek	124	6.4	0.6
	Kokee	124	17.6	1.6
	NRAO20	124	12.1	1.1
	Wettzell	124	13.0	1.2

*Standard deviation of a single height difference

**Standard deviation of the averaged height differences

Concluding remarks

- Station motions due to plate motions (PM) deform all tetrahedra. Dispersion for all configurations (corrected for PM) is 10^{-3} μstrain for the strain tensor elements, and a few mas for the rotations. Tensor elements show the presence of deformations as significant as rigid body motions which are not known in current reference frame solutions, including ITRF
- All tetrahedra solutions are free from translations because of the solution constraint
- PM corrected tetrahedral deformations are not cumulative

➔CONT.

Concluding remarks (cont.)

- Series for interval variations pass the randomness test. Also autocorrelations are less than the white noise level, therefore:
- The reference station heights at a reference epoch in one of these global vertical datums will be known within 0.5 – 3 mm. The station height uncertainties at an arbitrary epoch will vary within 10 – 35 mm.
- Despite the random nature of the height variations, only partial improvements are possible using polyhedral solutions. We have shown that the cyclic variations exist below noise level and they do not die down with the increasing lag (possibly due to earth tides and tidal loading at near coastal stations, and residual atmospheric effects on VLBI baselines)