BLIMPBE and its Geodetic Applications

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Abstract. Adjustment of geodetic data using coordinate-based formulations leads to rank-deficient Gauss-Markov models depending on the observation type. Among alternatives, minimum norm least squares solution (equivalently BLUMBE) is used to overcome this deficiency. In this study we examine further extensions of the minimum norm solution in the least-squares solution space, such as partial MINOLESS, which make use of a selection matrix. We derive Best Linear Minimum Partial Bias Estimation (BLIMPBE) via bias minimization from which partial MINOLESS can be obtained as a special case. We show, through an example, that BLIMPBE can be used effectively to control the contribution of the various parameters to the overall solution in the presence of model or observation biases.

Keywords. Rank-deficient Gauss-Markov model, Minimum Norm Least Square Solution MINOLESS

Introduction

Depending on the observation type, adjustment of geodetic data using coordinate-based formulations leads to rank-deficient Gauss-Markov models. Among alternatives, minimum norm least squares solutions are used to overcome this deficiency. A typical example, also important in its own right, is the realization of a terrestrial reference frame from modern space geodetic methods such as Satellite Laser Ranging (SLR), Very Long Baseline Interferometry (VLBI), Satellite Altimetry (SA), and Global Positioning System (GPS). All these methods necessitate the definition of a Terrestrial Reference System (TRS). The realization of such a system is achieved through a set of station coordinates, which are determined by the space geodetic methods, and the parameters and constants used in this process. The resulting realization is termed Terrestrial Reference Frame, (TRF). Among the alternatives, the realization of a TRS at a given epoch can be achieved by estimating the coordinates of a number of stations generating the observational data such as baselines, from a rank-deficient Gauss-Markov model using MINOLESS (Iz and Eubanks, 2000).

In this process, it is desirable to monitor and control the impact of individual stations to the overall adjustment of the data and estimation of station coordinates. If, for instance, one of the coordinate components of a station will not be well-defined because of a model deficiency or measurement error, the contribution of this station coordinate component to the overall definition of the reference frame should be reduced.

In the following sections, we first examine an extension of the minimum norm solution in the least-squares solution space, namely partial MINOLESS which makes use of a selection matrix with the above purpose in mind. We derive *Best Linear Minimum Partial Bias Estimation* (BLIMPBE) via bias minimization from which partial MINOLESS is obtained as a special case. We then show, using real data, that BLIMPBE can be used effectively to control the influence of various parameters to the overall solution in the presence of model or observation biases.

The rank-deficient Gauss-Markov Model

Consider the following linear model

$$y = \underset{n\times m}{A}\xi + e \qquad e \sim (0, \sigma_0^2 \underset{n\times n}{P^{-1}})$$

where

$$rkA =: q < \min\{m, n\}$$

In the above expression P is a symmetric positive definite matrix. The terms in the parentheses are the expected value and the variance/covariance matrix of the disturbances (observational errors).

LEast-Squares Solution (LESS) Space

The least-squares solution space to the rank deficient Gauss-Markov model is given by

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$$\{\hat{\xi} \mid N\hat{\xi} = c,\ [N,c] := A^T P[A,y]\} =$$
$$= \hat{\xi}_{special} + nullspace(A) =$$
$$= \{N^- c \mid N^- g - \text{inverse of N}\} =$$
$$= \{N^-_{rs} c \mid N^-_{rs} \text{ reflexive symmetric g-inverse of } N\}$$

where

$$N^- \text{g-inverse iff } NN^-N = N,$$
$$N^- \text{reflexive iff } N^-NN^- = N^-,$$
$$N^- \text{symmetric iff } N^- = (N^-)^T.$$

Note that there exist non-symmetric g-inverses N^- of the symmetric matrix N. The g-inverse N^- is reflexive if and only if $rkN^- = rkN = q$. All the reflexive symmetric g-inverses of the symmetric positive-semidefinite matrix N are positive-semidefinite. However, non-reflexive symmetric g-inverses of N may or may not be positive-semidefinite.

The MInimum NOrm LEast-Squares Solution

The minimum norm least-squares solution (MINOLESS) for the rank deficient Gauss-Markov model via

$$\|\hat{\xi}\|^2 = \hat{\xi}^T \hat{\xi} = \min_{\hat{\xi}}\{N\hat{\xi} = c\}$$

gives

$$\hat{\xi}_{MINOLESS} = N(NN)^- c = [N(NN)^- N(NN)^- N]c$$
$$= N(NNN)^- Nc$$

which is independent of the chosen g-inverse $(NN)^-$, respectively $(NNN)^-$.

$$\boxed{\hat{\xi}_{MINOLESS} = N^+ c}$$

where N^+ is pseudo inverse of N iff

(1) $NN^+N = N$,

(2) $N^+NN^+ = N^+$,

(3) NN^+ is symmetric,

(4) N^+N is symmetric.

N^+ always exists and unique.

Alternative principle of Best Linear Uniformly Minimum Bias Estimation

The estimation is *linear*, i.e.

$$\hat{\xi} = Ly$$

with *uniformly minimum bias* where the minimum bias vector

$$\beta = E\{\hat{\xi}\} - \xi = (LA - I_m)\xi$$

gives

$$tr(LA - I_m)(LA - I_m)^T = \min; \Rightarrow AA^T L^T = A.$$

Best or minimum mean squared error is obtained from

$$tr\, MSE\{\hat{\xi}\} = tr\, D\{\hat{\xi}\} + \beta\beta^T$$
$$= \sigma_0^2\, tr\,(LP^{-1}L^T) + \beta\beta^T$$
$$= \min_L\{AA^T L^T = A\}$$

which gives

$$\boxed{\begin{array}{l} \hat{\xi}_{BLUMBE} = N^+ c = \hat{\xi}_{MINOLESS} \\ \left.\begin{array}{l} D\{\hat{\xi}\} = \sigma_0^2 N^+ \\ \beta = -(I_m - N^+N)\xi \end{array}\right\} \Rightarrow MSE(\hat{\xi}) = \sigma_0^2 N^+ + \beta\beta^T \end{array}}$$

$$\boxed{\begin{array}{l} \hat{\sigma}_0^2 = (n-q)^{-1}(y^T P y - c^T \hat{\xi}_{BLUMBE}), \\ \hat{D}(\hat{\sigma}_0^2) = 2(\hat{\sigma}_0^2)^2 (n-q)^{-1} \end{array}}$$

(cf., e.g., E. Grafarend/B. Schaffrin 1993, Theorem 2.14 and Theorem 2.28).

Partial MINOLESS

Partial MINOLESS satisfies the following condition

$$\|\hat{\xi}\|_S^2 = \hat{\xi}^T S \hat{\xi} = \min_{\hat{\xi}}\{N\hat{\xi} = c\}$$

with

$$S := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

being *a selection matrix* such that $S + N$ is invertible *i.e.* $rk\, S \geq m - q$. Minimization gives

$$\hat{\xi} = (S+N)^{-1} N[N(S+N)^{-1}N]^{-} c = \\ = \{(S+N)^{-1} N[N(S+N)^{-1}N]^{-} N[N(S+N)^{-1}N]^{-} \\ N(S+N)^{-1}\} c$$

Note that the partial MINOLESS is no longer uniformly minimum biased because

$$AA^T\{PA[N(S+N)^{-1}N]^{-}N(S+N)^{-1}]\neq A.$$

Partially Best LUMBE

The partially best linear minimum bias estimator is obtained from

$$\sigma_0^2 tr(SLP^{-1}L^T) + \beta\beta^T = \min_L\{AA^T L^T = A\}$$

$$\Rightarrow SL = SA^T(ANA^T)^{-} AA^T P = \\ = SN(NNN)^{-} NA^T P = \\ = SN^+ A^T P$$

This result indicates that $\hat{\xi}_{BLUMBE} = N^+ c$ already minimizes any part of the trace of $MSE\{\hat{\xi}_{BLUMBE}\}$ — along with its total trace— within the class of LUMBEs.

Nevertheless, $\hat{\xi}_{BLUMBE}$ may not be the only solution which minimizes part of the $MSE\{\hat{\xi}\}$ in this class.

Best Linear Minimum Partial Bias Estimation (BLIMPBE)

The best linear minimum partial bias estimation is *linear*, i.e.

$$\hat{\xi} = Ly$$

with minimum partial bias

$$tr(LA - I_m)\xi\xi^T(LA - I_m)^T = \min_L \text{ for all } \xi \in \Re(\bar{S})$$

$$\Rightarrow (A\bar{S}A^T)L^T = A\bar{S}$$

with a suitable positive-semidefinite (e.g. *selection*) matrix.

The estimator should also be *best* in the sense of *minimum mean squared error,* i.e.

$$tr\ MSE\{\hat{\xi}\} = \sigma_0^2 tr(LP^{-1}L^T) + \beta\beta^T \\ = \min_L\{(A\bar{S}A^T)L^T = A\bar{S}\}.$$

We define the following *Lagrange* target function

$$\Phi(L^T,\Lambda) := tr\{(LP^{-1}L^T) - 2\Lambda^T[(A\bar{S}A^T)L^T - A\bar{S}\} \\ = \underset{L^T,\Lambda}{\text{stationary}}$$

with the *Euler-Lagrange* necessary conditions

$$\frac{1}{2}\frac{\partial\phi}{\partial L^T} = P^{-1}L^T - (A\bar{S}A^T)\Lambda \doteq 0 \quad (1)$$

$$\frac{1}{2}\frac{\partial\phi}{\partial\Lambda} = A\bar{S} - (A\bar{S}A^T)L^T \doteq 0 \quad (2)$$

and sufficient condition

$$\frac{1}{2}\frac{\partial^2\phi}{(\partial vecL^T)(\partial vecL^T)^T} = \frac{\partial vec(P^{-1}L^T)}{(\partial vecL^T)^T} = I_m \otimes P^{-1}\ p.d.$$

The solution to the *Euler-Lagrange* conditions is given by

$$\hat{\xi}_{BLIMPBE} = [\bar{S}N(N\bar{S}N\bar{S}N)^{-} N\bar{S}]^{-} c$$

with

$$D\{\hat{\xi}_{BLIMPBE}\} = \sigma_0^2[\bar{S}N(N\bar{S}N\bar{S}N)^{-} N\bar{S}]$$

which is independent of the g-inverse $(N\bar{S}N\bar{S}N)^{-}$. The means square error matrix of the estimates is given by

$$MSE\{\hat{\xi}\} = \sigma_0^2[\bar{S}N(N\bar{S}N\bar{S}N)^{-} N\bar{S}] + \beta\beta^T$$

where the *bias* vector β is

$$\beta = -[I_m - \bar{S}N(N\bar{S}N\bar{S}N)^{-} N\bar{S}N]\xi$$

Also

$$\hat{\sigma}_0^2 = (n-q)^{-1}(y^T Py - c^T\hat{\xi}_{BLIMPBE}), \\ \hat{D}\{\hat{\sigma}_0^2\} = 2(\hat{\sigma}_0^2)^2(n-q)^{-1}$$

(cf., E. Grafarend/B. Schaffrin 1993, chapter 2(c) for further representations).

Note

Our notation of $\bar{S}$ indicates the choice of a selection matrix, but any other symmetric positive-semidefinite matrix should be fine.

The particular choice of $\bar{S} := (S+N)^{-1}$ where S denotes the selection matrix of the Partial MINOLESS above suggests its interpretation as BLIMPBE due to the identity

$$\hat{\xi}_{BLIMPBE} = (S+N)^{-1}N[N(S+N)^{-1}N(S+N)^{-1}N]^{-} N(S+N)^{-1}c$$
$$= (S+N)^{-1}N[N(S+N)^{-1}N(S+N)^{-1}N]^{-}c$$
$$= \hat{\xi}_{PMINOLESS}$$

Numerical Example and Conclusion

As an example, we use *simultaneous* VLBI baselines generated by the USNO 1999-3 solution for an operational realization of terrestrial reference frames. We first remove the effect of station motions due to the plate motion onto the baselines, then establish a rank-deficient Gauss-Markov model[1].

Since only baselines are used as observables, the deficiency in rank for the reference frame is six (three rotations and three shifts).

We derive three-dimensional coordinates for three different fundamental tetrahedrons, formed by the combination of four VLBI stations each, namely:Fortaleza—Kokee—NRAO20—Wettzell, Gilcreek—Hartrao—Matera—Westford, and Gilcreek—Kokee-NRAO20—Wettzell as shown in Figures 1–3.

In this process, we perform a number of free network solutions referenced to a common set of adopted nominal coordinates for VLBI stations using simultaneous observations at different epochs. The adjusted coordinates of VLBI stations located at the apexes of the three tetrahedrons are first calculated using MINOLESS.

Table 1. Position differences between MINOLESS (M) and BLIMPBE (Bb) and MINOLESS (Mb) with biased setup for the fist stations in each tetrahedron.

Tetrahedron	Median distance in mm		
	Mb - M	Bb - M	Expected
FKNW	25	143	87
GHMW	10	94	87
GKNW	6	65	87

We subsequently introduce a bias into the position of the first station of each tetrahedron (50 mm to each coordinate component)—thereby biasing also the corresponding baselines observed from each of these stations by approximately 87 mm. We then re-estimate the station coordinates, this time using MINOLESS and BLIMPBE. In the case of BLIMPBE we use the inverse of three times the bias error for the biased station in the selection matrix. To assess how the error at a single station is handled by the adjustment method, we calculate the differences between the adjusted positions by BLIMPBE and MINOLESS with *biased* setup as well as MINOLESS positions using *unbiased* setup. Note that, in all cases, three additional elements of the selection matrix in BLIMPBE are modified to account for the remaining three rank deficiencies.

Figure 4 shows the norm of the estimated corrections in comparison to the nominal values of the first stations of tetrahedrons calculated using MINOLESS as well as BLIMPBE and MINOLESS with biased setup. Calculations are repeated for simultaneous data available at different epochs but the coordinate biases are kept the same.

Observe that for different tetrahedrons and for alternative data sets, the norm of the MINOLESS corrections are smaller than the BLIMPBE's norms regardless whether biased setup is used or not. This is a result of the intrinsic property of the MINOLESS solution where the smallest norm for the estimated parameters is assured. Although this is a desirable property as long as the underlying model assumptions are fulfilled, it underestimates the parameters if model or observations are biased. In this case, BLIMPBE is more effective than the MINOLESS, as shown in Table 1, in correcting the estimates to account for the biases.

The Table 1 results also show that the median differences of biased setup solutions for BLIMPBE are closer to the expected differences due to the artificially induced biases.

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References

Grafarend E., and B. Schaffrin (1993) Adjustment Computations in Linear Models (in German), Bibliograph.Inst.: Mannheim, 483 pg.

Iz, H.B., and M. Eubanks (2000) Space-time Deformations of Global Tetrahedra from USNO Baselines and their Contributions to the Reference Frame Realization, Abstracts of EGS, XXV General Assembly, Millennium Conference on Earth, Planetary and Solar System Sciences, Nice, France.

[1] Note that the mathematical model—the distance between two stations—is non-linear. It is linearized for the solutions and the established linear estimation theory is applied to the linearized model. Solutions are not iterated to preserve a consistent set of adjusted station coordinates. Otherwise each iteration leads to a different set of station coordinates referring to different nominal values for each solution changing from epoch to epoch. The lack of iteration does not pose a problem as long as the stations' nominal coordinates are defined as close as possible to their true values.

Figure 1. FKNW tetrahedron stations: Fortaleza, Kokee, NRAO20, Wettzell.

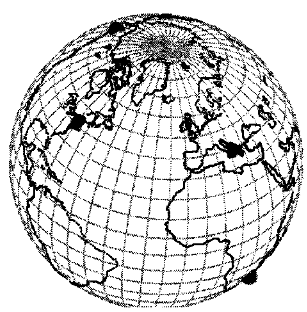

Figure 2. GHMW tetrahedron stations: Gilcreek, Hartrao, Matera, Westford.

Figure 3. GKNW tetrahedron stations Gilcreek, Kokee, NRAO20, Wettzell.

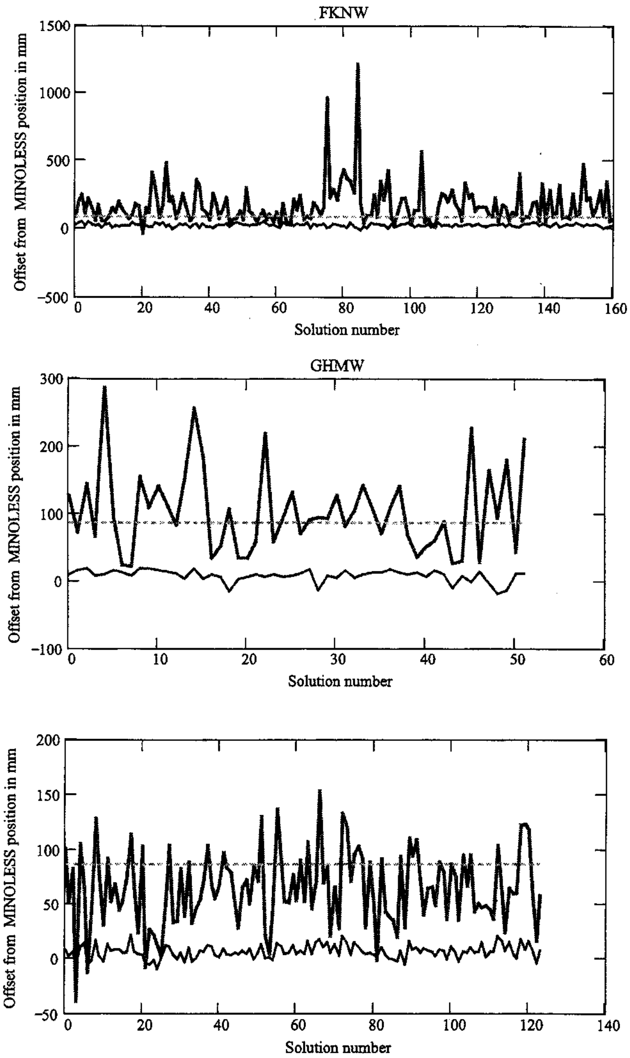

Figure 4. Solutions for a series of baseline data for each tetrahedron. Bold trace is from the BLIMPBE solutions with biased setup, dashed trace is the expected offset from the MINOLESS with unbiased setup, and thin trace is from the MINOLESS with biased setup. Offsets are for the first stations of each tetrahedron.