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COMPARISON OF FOUR GEODETIC NETWORK DENSIFICATION SOLUTIONS

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ABSTRACT

Geodetic surveying practices require the establishment of new control stations that are tied to higher order stations with known coordinates (reference control stations – fiducial stations), an activity known as densification. The coordinates of the newly established control stations are adjusted together with the coordinates of the reference control stations which are required to remain invariant along with their variance-covariance matrices (i.e. they are reproduced). In this study, we compare the performance of four different approaches that can be used to achieve this end for their optimality and sensitivity against potential systematic effects in the reference control station coordinates using a GPS network densification example.

KEYWORDS: Geodetic network. Densification. Control stations. GPS network. Systematic effects.

INTRODUCTION

Geodetic control has been of fundamental importance to various activities such as geodetic, geophysical, surveying and civil engineering. With a view to developing a proper geodetic control, several studies concentrated on numerical methods such as minimal constraints [**7**] [**8**] [**12**], free net (inner constraints) [**4**] [**7**] [**14**] [**15**] and unbiased free net adjustment [**7**] [**8**] [**21**] within the framework of geodetic datums (zero order design). A comprehensive review of the past theoretical developments in this area can be found in [**22**] and [**6**].

The above concepts have been extended and brought into the geodetic network densification problem where the application of stochastic and non-stochastic constraints in the calculation of the control station coordinates has been discussed by various investigators [**1**] [**5**] [**13**] [**16**] [**17**]. In the establishment of new control stations which are tied to higher order stations with known coordinates (reference control stations – fiducial stations), the coordinates of the new stations are adjusted together with the coordinates of the reference control stations which are preferably kept invariant along with their variance-covariance matrices. If the solution results for the reference control station coordinates remain unchanged, the solution is called *reproducing*, otherwise, it is *non-reproducing*. Of course, a non-reproducing solution is applicable if the re-adjusted reference control station coordinates and their variance-covariance matrix are ignored.

From the numerous solution formulations, we have examined four different adjustment methods, namely:

1. Densification with S*tochastic Constraints* (SC) is a well known least-squares adjustment procedure where the reference control station coordinates are adjusted along with their previously obtained covariance matrix as (weak) constraints together

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with the relevant geodetic measurements. The solution gives the Best Linear Uniformly Unbiased Estimate, BLUUE, for the unknown parameters within this model while the reference control station coordinates are adjusted simultaneously and, hence, are subject to change. This solution is therefore a *non-reproducing* one, also called a *dynamic solution*. The adjusted reference station coordinates, however, are not kept static over time. A variant of this approach is discussed in the context of collocation in [**5**]

2. Densification with *Seemingly Non-Stochastic Constraints* (SNSC) is an adapted version of the well-known solution with fixed constraints (which is, of course, not considered in this study). The non-stochastic constraints are utilized to accommodate the weight matrix of the reference control station coordinates, which is available a priori in the adjustment, for the computation of the covariance matrix of the adjusted station coordinates [**13**]. The solution is a *Linear Uniformly Unbiased Estimator* with reproducing property, repro-LUUE, i.e. it reproduces the reference control station coordinates and produces linear uniformly unbiased estimates of the densification control station coordinates. The estimates are nonetheless not the best in the class of reproducing unbiased estimates [**19**].

3. Another densification solution, which is tailored to reproduce the reference control station coordinates along with their covariance matrix directly, has recently been developed by one of the authors, and is known as the *Best Linear Uniformly Unbiased Estimation with Reproducing Property,* repro-BLUUE [**19**]. In addition to its reproducing property, this estimator gives the BLUUE for the new control station coordinates [**18**] [**19**].

4. Finally, we discuss the densification with a free adjustment which is followed by a similarity (Helmert) transformation. This method is a frequently used in GPS type network densification [**19**]. The solution reproduces reference station coordinates and generates control station coordinate estimates that are BLUUE. We will abbreviate this solution as Helmert-LUUE.

This paper aims to quantify the performance of these approaches in a GPS network densification example. We will first summarize the essential operational relationships for the different solution methods and subsequently apply them to a GPS densification problem for numerical comparison under different scenarios that may occur in practice.

FORMULATIONS OF THE ALTERNATIVE SOLUTION METHODS

Stochastic Model

We begin with a definition of the stochastic model that, after linearization, reads as

$$y = \underset{n\times\ell}{A_1}\,\xi_1 + \underset{n\times(m-\ell)}{A_2}\,\xi_2 + e, \quad rk\,A_2 = m-\ell < q := rk[\mathrm{A}_1, \mathrm{A}_2], \tag{1}$$

$$\hat{\xi}_1 = \xi_1 + e_1^0 = K\xi + e_1^0, \quad K := [I_\ell, 0], \tag{2}$$

$$\begin{bmatrix} e \\ e_1^0 \end{bmatrix} \sim (\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & Q_1^0 \end{bmatrix}), \tag{3}$$

where y is the $n\times 1$ vector of observational increments to the nominal values of the observations, calculated using the observations and the approximate coordinate values for the unknown control station coordinates and the known coordinates of the reference stations. ξ_1 is the $l \times 1$ vector of reference station coordinate increments to the nominal (approximate) values of unknown coordinates. ξ_2 is the $(m-l) \times 1$ vector of densification station coordinate increments with respect to their approximate values, all to be estimated. $\mathrm{A} := [\mathrm{A}_1, \mathrm{A}_2]$ is the $n \times m$ coefficient (design) matrix, e is the $n \times 1$ vector of random observation errors, $\hat{\xi}_1$ is the $l \times 1$ vector of reference control station coordinates (known), e_1^0 is the corresponding $l \times 1$ vector of random errors of the a priori reference control station coordinates, σ_0^2 is the unknown variance of unit weight, P is the $n \times n$ weight matrix of the observations, Q_1^0 is the $l \times l$ cofactor matrix of the reference control station coordinates.

Here, we assume that the known coordinate information of the control stations is unbiased, and the variance of unit weight is the same for both the a priori and the observational information.

The Densification Solution with Stochastic Constraints (SC)

In the above model (1) to (3), the Best Linear Uniformly Unbiased (non-reproducing) Estimate (BLUUE) of ξ is given by

$$\hat{\hat{\xi}} = (N + K^T P_1^0 K)^{-1} (c + K^T P_1^0 \hat{\xi}_1) \tag{4}$$

if $P_1^0 = (Q_1^0)^{-1}$ exists, with $\left[N, c\right] = A^T P \left[A, y\right]$. Then the variance-covariance matrix of $\hat{\hat{\xi}}$ is

$$D\{\hat{\hat{\xi}}\} = \sigma_0^2 (N + K^T P_1^0 K)^{-1}. \tag{5}$$

The residual vector and the estimated variance of unit weight are provided through:

$$\tilde{\tilde{e}} = y - A_1 \hat{\hat{\xi}}_1 - A_2 \hat{\hat{\xi}}_2 = y - A \hat{\hat{\xi}} \tag{6}$$

$$\tilde{\tilde{e}}_1^0 = \hat{\xi}_1 - \hat{\hat{\xi}}_1 = \hat{\xi}_1 - K \hat{\hat{\xi}} \tag{7}$$

$$\hat{\hat{\sigma}}_0^2 = \frac{\tilde{\tilde{e}}^T P \tilde{\tilde{e}} + (\tilde{\tilde{e}}_1^0)^T (Q_1^0)^{-1} \tilde{\tilde{e}}_1^0}{n - m + l} \tag{8}$$

Corrected Densification Solution with Seemingly Non-Stochastic Constraints (SNSC)

Also known as the *fiducial network* strategy deployed by JPL, it is an adjustment in which the formula for non-stochastic constraints is applied, but after the variance-covariance matrix of the reference stations is introduced into the formulation for error propagation, and thus for calculating the variance-covariance matrix of the adjusted densification control station coordinates.

The estimate of ξ is given by

$$\begin{aligned}\breve{\bar{\xi}} &= (N + K^T P_1^0 K)^{-1} c \\ &+ (N + K^T P_1^0 K)^{-1} K^T \cdot \\ &[K (N + K^T P_1^0 K)^{-1} K^T]^{-1} [\hat{\xi}_1 - K (N + K^T P_1^0 K)^{-1} c].\end{aligned} \tag{9}$$

The following variance-covariance matrix of $\breve{\bar{\xi}}$ reflects fully the impact of the a priori variance-covariance matrix used in conjunction with the formulas for the non-stochastic constraints:

$$
\begin{aligned}
D\{\breve{\bar{\xi}}\} &= \sigma_0^2 (N + K^T P_1^0 K)^{-1} \\
&\quad - \sigma_0^2 (N + K^T P_1^0 K)^{-1} K^T \cdot \\
&\quad [K(N + K^T P_1^0 K)^{-1} K^T]^{-1} K(N + K^T P_1^0 K)^{-1} \\
&\quad + \sigma_0^2 (N + K^T P_1^0 K)^{-1} K^T \cdot \\
&\quad [K(N + K^T P_1^0 K)^{-1} K^T P_1^0 K(N + K^T P_1^0 K)^{-1} K^T]^{-1} \cdot \\
&\quad K(N + K^T P_1^0 K)^{-1}.
\end{aligned} \tag{10}
$$

The residual vector and the estimated variance of unit weight are provided through the following expressions:

$$\breve{\bar{e}} = y - A\breve{\bar{\xi}} \tag{11}$$

$$\breve{\bar{\sigma}}_0^2 = \frac{\breve{\bar{e}}^T P \breve{\bar{e}}}{n - m + tr\left[K(N + K^T P_1^0 K)^{-1} K^T P_1^0\right]^{-1}} \tag{12}$$

Best Linear Uniformly Unbiased Estimator with Reproducing Property repro-BLUUE

The best linear uniformly unbiased estimate of ξ that reproduces $\hat{\xi}_1$ is given by

$$\bar{\xi} = \hat{\hat{\xi}} + K^T (KK^T)^{-1} \tilde{\tilde{e}}_1^0. \tag{13}$$

This solution method is based on the idea of recovering the original values of the adjusted reference station coordinates [**19**]. The variance-covariance matrix of $\bar{\xi}$ for the control station coordinates and the unknown densification control station coordinates are given by

$$D\{\bar{\xi}_1\} = D\{\hat{\xi}_1\} = \sigma_0^2 Q_1^0 \tag{14}$$

$$D\{\bar{\xi}_2\} = \sigma_0^2 [N_{22} - N_{21}(N_{11} + P_1^0)^{-1} N_{12}]^{-1} \tag{15}$$

The residual vector and the estimated variance of unit weight are provided through:

$$\bar{e} = y - A\bar{\xi} \tag{16}$$

$$\bar{\sigma}_0^2 = \frac{\bar{e}^T P \bar{e}}{n - m + l + tr(I_l + N_{11} Q_1^0) S_1 (S_1 + P_1^0)^{-1}} \tag{17}$$

Inner Constraints (Free) Adjustment Followed by a Similarity Transformation (Helmert-LUUE)

In this approach, the solution estimate is obtained using inner constraints which are expressed through a $(m-q)\times n$ matrix E with the following properties [**11**]:

$$AE^T = 0, \quad \text{rk A} + \text{rk E} = \text{m}, \quad \text{q} := \text{rk A}, \tag{18}$$

$$z_0 := EK^T (KK^T)^{-1} \hat{\xi}_1 = \bar{K}\xi + e_0^0, \tag{19}$$

$$\bar{K} := EK^T (KK^T)^{-1} K, \quad rk\, \bar{K} = m - rk\, \text{A} = \text{m - q}, \tag{20}$$

$$e_0^0 := EK^T (KK^T)^{-1} e_1^0 \sim (0, \sigma_0^2 Q_0^0), \quad \text{C}\{\text{e}_0^0, e\} = 0, \tag{21}$$

$$Q_0^0 := EK^T (KK^T)^{-1} Q_1^0 (KK^T)^{-1} KE^T, \quad P_0^0 := (Q_0^0)^{-1}, \tag{22}$$

and where $K^T (KK^T)^{-1} K = K^T K$ represents a *selection matrix*.

The optimal estimate, which is independent of P_0^0, is given by

$$
\begin{aligned}
\bar{\bar{\xi}} &= (N + \bar{K}^T P_0^0 \bar{K})^{-1} (c + \bar{K}^T P_0^0 z_0) \\
&= (N + \bar{K}^T \bar{K})^{-1} (c + \bar{K}^T z_0)
\end{aligned} \tag{23}
$$

with the following residual vector of the reference control station coordinates

$$\overline{\overline{e}}_1^0 = \hat{\xi}_1 - \overline{\overline{\xi}}_1 = \hat{\xi}_1 - K\overline{\overline{\xi}} \neq 0. \tag{24}$$

Since $\overline{\overline{e}}_1^0$ does not vanish, in general, the estimate $\overline{\overline{\xi}}$ will not guarantee to reproduce the reference control station coordinates. The estimate of ξ, which guarantees to reproduce the original reference control station information along with their variance-covariance matrix is given by

$$\hat{\overline{\overline{\xi}}} = \overline{\overline{\xi}} + K^T(KK^T)^{-1}\overline{\overline{e}}_1^0 \tag{25}$$

together with the following variance-covariance matrix of $\hat{\overline{\overline{\xi}}}$ for the reference control station coordinates and the unknown densification station coordinates

$$D\{\hat{\overline{\overline{\xi}}}_1\} = \sigma_0^2 Q_1^0, \tag{26}$$

$$D\{\hat{\overline{\overline{\xi}}}_2\} = \sigma_0^2[N_{22} - N_{21}(N_{11} + E_1^T P_0^0 E_1)^{-1} N_{12}]^{-1}, \tag{27}$$

where

$$N + \overline{K}^T P_0^0 \overline{K} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} + \begin{bmatrix} E_1^T P_0^0 E_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{28}$$

and

$$E_1 := EK^T. \tag{29}$$

The residual vector and the estimated variance of unit weight are obtained from:

$$\hat{\overline{\overline{e}}} = y - A\hat{\overline{\overline{\xi}}} \tag{30}$$

$$\hat{\overline{\overline{\sigma}}}_0^2 = \frac{\hat{\overline{\overline{e}}}^T P \hat{\overline{\overline{e}}}}{n - q + E\{\tilde{\overline{\overline{R}}}\}} \tag{31}$$

where

$$E\{\tilde{\overline{\overline{R}}}\} = tr\left\{N_{11}[I_l - KE^T(\overline{K}E^T)^{-1}EK^T]Q_1^0 \\ [I_l - KE^T(\overline{K}E^T)^{-1}EK^T]\right\} + trN_{11}(KN_{rs}^- K^T) \tag{32}$$

and

$$N_{rs}^- := (N + \overline{K}^T P_0^0 \overline{K})^{-1} N (N + \overline{K}^T P_0^0 \overline{K})^{-1}. \tag{33}$$

DENSIFICATION PROJECT DESCRIPTION

A control network for an engineering site is shown in Figure 1. The network includes three reference stations with known reference station coordinates in the WGS84 coordinate system; here HKKT (Hong Kong Kam Tin), HKFN (Fanling) and HKLT (Lam Tei) serve as reference control stations. GPS was the survey method during the 2/9/2004 and 10/9/2004 one-day campaigns. Full two days of GPS raw observation data were processed using the Topcon Pinnacle software to calculate the 22 baseline vectors shown in Figure 1, and their covariance matrices.

In the above network, the minimum and maximum baseline lengths and their standard deviations are 343.428±0.001 m and 9827.415±0.011 m respectively. We assumed that the standard deviations of the reference station coordinates are ±0.01m for each XYZ component.

Fig. 2 displays the precision of the GPS baseline vector measurements. The clustering observed in

Fig. 2 is a result of the presence of very short and very long baseline lengths.

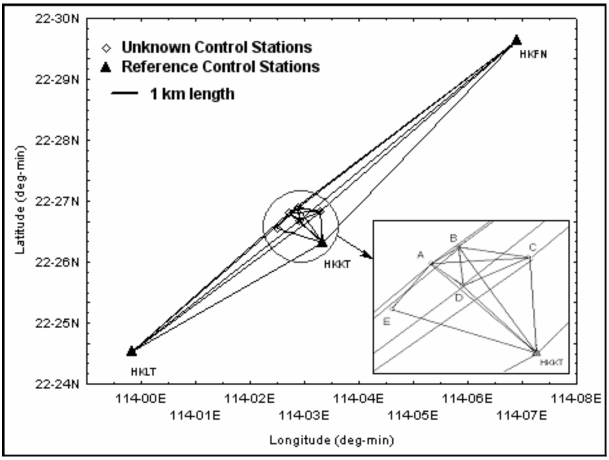

Fig. 1. Densification Control Network for an Engineering Project in Hong Kong.

NUMERICAL SOLUTIONS UNDER DIFFERENT SCENARIOS

Solutions with the Existing Data

We first obtained alternative solutions for estimating the coordinates of the densification control stations using the GPS baseline vector measurements.

Fig. 3 shows the a posteriori variance of unit weight, $\hat{\sigma}_0^2$, which we will use to assess the goodness-of-fit of the different solutions and to judge the appropriateness of the adopted scaling for our weight matrix.

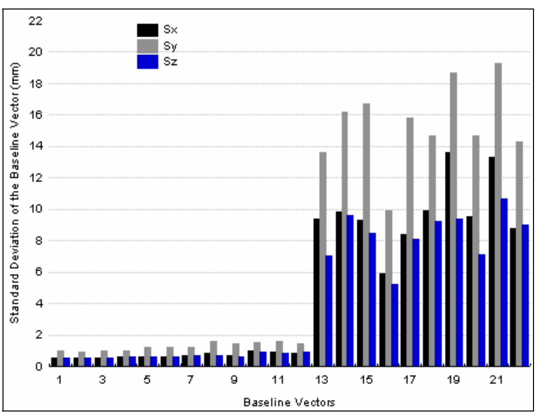

Fig. 2. Magnitudes of the precisions of the GPS baseline vector measurement ordered in increasing baseline lengths. The two clusterings are due to the very short and very long baseline lengths.

The $\hat{\sigma}_0^2$ of the SC solution, being nearly equal to one, indicates that the standard deviations of both the reference control stations and the calculated GPS baseline vector components, which are ±1 cm and ±(1 mm to few cm) respectively (see Figure 2), are consistent with their expected values. Compared with this SC solution, the $\hat{\sigma}_0^2$ of the repro-BLUUE and the Helmert-LUUE are comparatively smaller and the $\hat{\sigma}_0^2$ of the SNSC (larger).

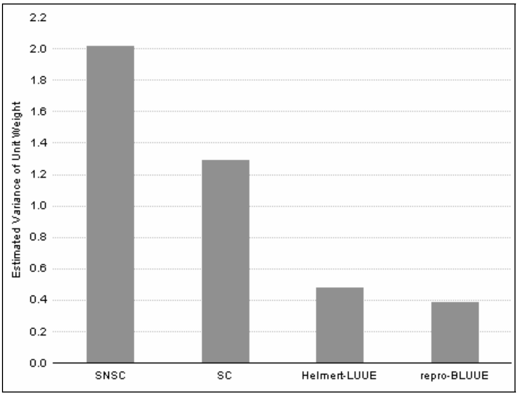

Fig. 3. The a posteriori variances of unit weight for alternative solutions.

The four estimators considered here yield unbiased results if the stochastic model (Eqs. 1 - 3) is correctly set up. Consequently, the mean squared error matrices coincide with the variance-covariance matrices which are now compared. Since all the solutions share the same geometry and observations, the variance-covariance matrices of the solutions are affected only by the formulations. Therefore, for a constant variance of unit weight, the trace of the variance-covariance matrix of the estimates (for the unknown station coordinates portion) provides information that results only from each solution's characteristics (Fig. 4). In this figure note that the traces of the variance-covariance matrices of the SC and repro-BLUUE solutions are exactly the same because the repro-BLUUE solution is actually derived from the SC solution with the addition of residuals for the recovery of the original precision of the reference stations.

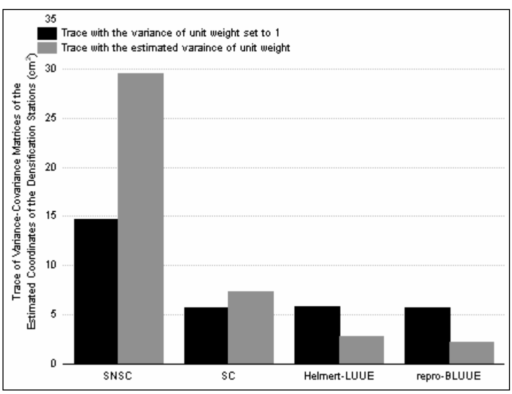

Fig. 4. Trace of the variance-covariance matrices of the estimated coordinates of the densification control stations with the (a priori) variance of unit weight set to one, and with the estimated (a posteriori) variance of unit weight.

More or less, the traces of the variance-covariance matrices of all four solutions are within an order of 2. If we consider the variance-covariance matrices of the solutions in the light of the estimated (a posteriori) variance of unit weight shown in

Fig. 3, we still observe a similar performance, with the repro-BLUUE solution showing the smallest trace (as it should).

We have also calculated, as shown in

Fig. 5, the trace of variance-covariance matrices for the adjusted observations, based on both the estimated (a posteriori) variance of unit weight and the (a priori) variance of unit weight set to one.

The larger trace value for the variance-covariance matrix of the observations of the SNSC solution is caused by the standardization of the non-stochastic constraints on the reference control station coordinates with a priori stochastic values, i.e. the solution is forced to follow the stochastic constraints *exactly*.

The trace value for the variance-covariance matrix of the observations of the SC solution is the smallest because its variance-covariance matrix of the estimates includes the precision of the reference station coordinates which are also adjusted in the solution. In contrast, both the repro-BLUUE and the Helmert-LUUE solution restore the uncertainty in the reference station coordinates in the process of recovering the reference station coordinates. This effect can also be seen in Eqs. (13) and (25).

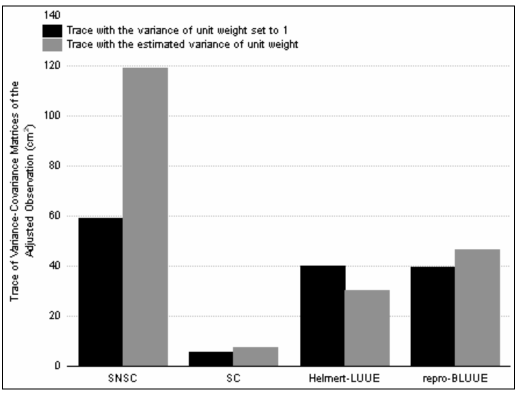

Fig. 5. Trace of the variance-covariance matrices of the adjusted baseline vector components with the a priori variance of unit weight set to one and with the estimated (a posteriori) variance of unit weight.

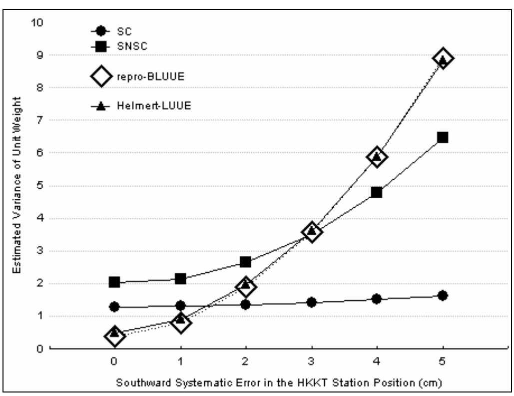

Fig. 6. Influence of a southward systematic error in the HKKT station position on the estimated (a posteriori) variance of unit weight. The std. deviations of the reference stations cords. are homogeneous and equal to $\pm 1cm$

These results reveal only the performance of the four alternative solutions when the stochastic model (i.e., Eqs. 1 - 3) is correct. In the following section, we examine their performance when this is not the case. We will assume that one of the reference station coordinates is affected by unknown systematic errors (i.e. biases).

Impact of unknown biases in the position of the HKKT reference control on the densification quality

We estimated the coordinates of the unknown control stations with the four different solution methods using a HKKT station position that is progressively falsified from 1 cm to 5 cm in the southerly direction. Under this scenario, the changes in the estimated (*a posteriori*) variance of unit weight, $\hat{\sigma}_0^2$, of each solution reveals the degree of sensitivity of the different approaches to erroneous reference control station information (i.e. coordinates along with variance-covariance matrices).

Fig. 6 shows an increasing trend in the estimated variances for all the solutions for increasing biases of the HKKT reference station position. The rate of change in the estimated variances of the SC solutions is, however, much smaller than for the others. This is due to the fact that this solution absorbs part of the bias in the position of the HKKT solution by adjusting it.

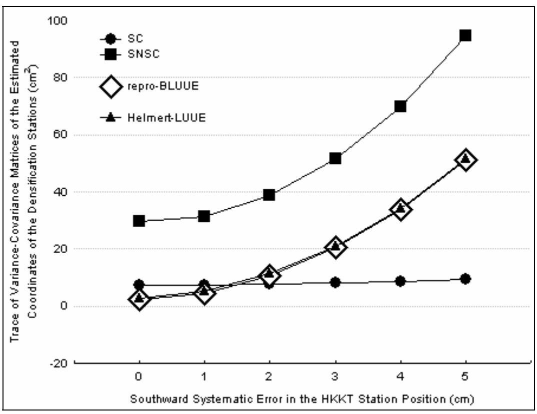

Fig. 7. Influence of a southward position error in the HKKT Station position on the trace of the variance-covariance matrices of the estimated coordinates of the densification stations (based on their respective estimated variances of unit weights.)

Impact of unknown biases in the position of the HKKT reference control on the precision and accuracy of the solutions

Fig. 7 shows the sensitivity of the trace of the variance-covariance matrix of the estimated densification control station coordinates for each solution method to a progressively increasing southward position error in the HKKT station. It displays the trace of the variance-covariance matrices of the estimated densification station coordinates, based on their respective estimated variances of unit weight. Note that there are two factors affecting the behavior of the traces: the influence of the estimated (*a posteriori*) variance of unit weight which was discussed above, and the structural sensitivity of the formulation. The rates of change in this figure for all the solutions are in agreement with the effect of the changes that were observed in the changes of the estimated (*a posteriori*) variance of unit weight (Fig.6).

This discussion, however, is incomplete because the trace of the estimated coordinates is about the precision and not about the accuracy of the solutions if the estimates are biased. For a better insight, we calculated the changes in the adjusted densification control station positions with the adjusted results of each approach where no systematic position error was introduced. Overall, the adjusted results without any systematic error differ slightly for the various solution methods. In

Fig. 8 we report the averaged position changes (over all the densification stations) for each solution with increasing magnitude of a southward position error. We observe that the SNSC solution transfers the systematic error directly to the adjusted densification station coordinates which is not a desirable property. After all, the process of adjustment is to distribute random observation errors to obtain optimal solutions for the problems at hand. The coordinates of the densification station coordinates obtained with the other methods are less affected by the systematic error in the HKKT coordinates.

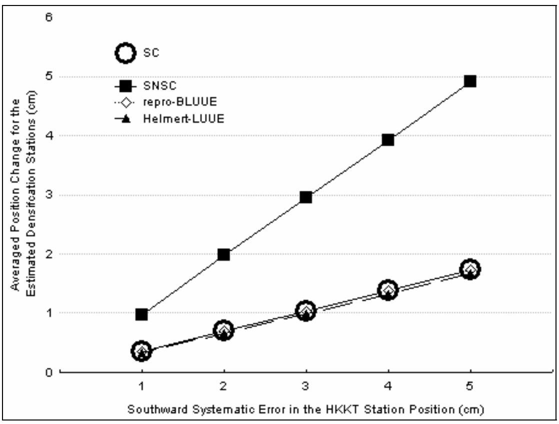

Fig. 8. Influence of a southward position error in the HKKT Station on the adjusted coordinates of the densification control stations. The standard deviations of the reference station coordinates are homogeneous and equal to ±1 cm.

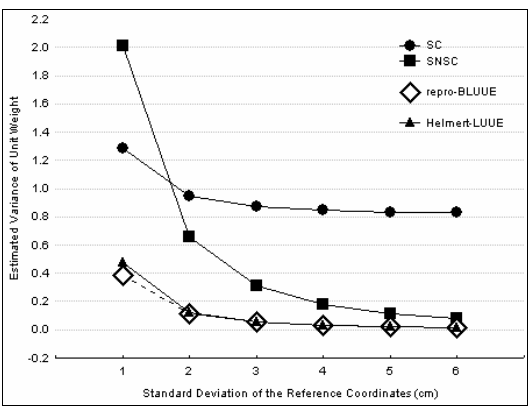

Fig. 9. Propagation of the reference station coordinates' precision (standard deviations) to the estimated (a posteriori) variance of unit weight of the solutions (to the goodness-of-fit). The position of the reference stations are the same in each case.

Propagation of the precision of the reference control station coordinate to the densification station coordinates

In this section we examine the response of the different densification methods to the precision of the reference control station coordinates. We calculated five additional solutions using standard deviations for the reference stations from ±1 cm to ±6 cm, in 1 cm increments.

Fig. 9. depicts the changes in the estimated (a posteriori) variances of unit weight for the different solution methods when the precision of the reference station coordinates is decreasing. In the SNSC solution, the exact constraints are relaxed to the extent allowed by the standard deviations of the reference station coordinates through standardization. The response of all the remaining methods is similarly stable for decreasing precision of the reference station coordinates and not markedly different from solution to solution.

We have also examined the changes in the trace of the variance-covariance matrices of the estimated coordinates of the densification control stations as a function of the standard deviations of the reference control stations (Fig.10)

Again, the response of the alternative solution methods is plausible as the SNSC solution takes full account of the precision of the reference station through standardization which explains the reason for the sharp increase with decreasing precision. All the other solution methods try to generate quasi-optimal solutions in adjusting the observations, thereby lowering their traces when compared with the SNSC results.

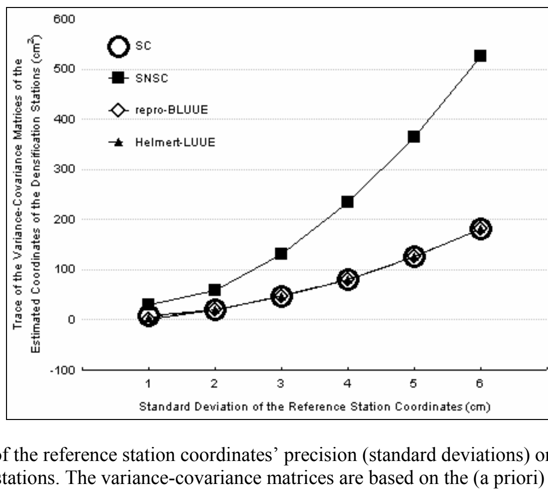

Fig. 10. Influence of the reference station coordinates' precision (standard deviations) on the precision of the densification control stations. The variance-covariance matrices are based on the (a priori) variance of unit weight being set to 1.

When there are no reliable information about the quality of the observations one uses the estimated (a posteriori) variance of unit weight in calculating the variance-covariance matrix of the adjusted densification control station coordinates instead of a chosen value. Under this scenario, the uncertainties of the adjusted station coordinates in the SC solution propagate by decreasing the precision of the reference station coordinates. This is a desirable property in densification, especially when we do not have reliable knowledge about the precision of the reference stations coordinates a priori. All the variance-covariance matrices of the remaining methods remain insensitive to this effect.

CONCLUSION

We have shown that the densification adjustment methods should be selected, depending on the quality of the reference station coordinates (whether or not they are influenced by systematic effects or noise), the quality of the baseline vector observations, and, more importantly, on how good the prior information about these parameters is. Our rather straight-forward comparison highlights the disadvantages of exact constraints, as the SNSC solution method fails to a certain extent in network densification, while all other methods shows similar performance. Since it is desired to keep the coordinates of the reference control stations along with their variance-covariance matrices unchanged after the adjustment, both repro-BLUUE and Helmert-LUUE could be utilized, particular in the maintenance of the original information of the control stations. This result may not be entirely new for seasoned practitioners, but it appears to be a forgotten wisdom in current practice, especially for those who are not intimately familiar with traditional geodetic applications.

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References

1. Aduol, F. W. O., 1993. A static-dynamic model for densification of geodetic networks. *Allgemeine Vermessungs-Nachrichten Intl. Edition*, **10**: 20-36.
2. Ananga, N., 1991. Least squares adjustments of seasonal leveling. *Journal of Surveying Engineering*, **117** (2): 67-76.
3. Ananga, N., Coleman, R., and Rizos, C., 1994. Zero-epoch heights and height velocities. *Survey Review*, **32** (251): 307-313.
4. Blaha, G., 1982. Free networks: Minimum norm solution as obtained by the inner adjustment constraint method. *Bulletin Géodésique*, **56**: 209-219.
5. Buiten, H. J., 1978. Junction of nets by collocation, *Manuscripta. Geodaetica.* **3**: 253-297.
6. Caspary, W. F., 1988. *Concepts of Network and Deformation Analysis*. Monograph 11, School of Surveying, The University of New South Wales, Kensington, N.S.W. Australia. 183 pages.
7. Grafarend, E., and Schaffrin, B., 1974. Unbiased free net adjustment. *Survey Review*, **22** (171): 200-218.
8. Grafarend, E., and Schaffrin, B., 1976. Equivalence of estimable quantities and invariants in geodetic networks. *Zeitschrift für Vermessungswesen* **101**: 485-491.
9. Grafarend, E., and Schaffrin, B., 1988. From the static to the dynamic treatment of geodetic networks (in German). *Zeitschrift für Vermessungswesen* **113**: 79-103.
10. Grimm-Pitzinger, A., and Hanke, K., 1990. Restrictions in geodetic networks. *Journal of Surveying Engineering*, **116** (3): 149-154.

11. Koch, K. R., 1983. The choice of the datum of a trigonometric network for the determination of new points (in German). *Zeitschrift für Vermessungswesen*, **108**: 104-111.
12. Leick, A., 1982. Minimal constraints in two-dimensional networks. *Journal of Surveying Engineering*, **108** (2): 53-68.
13. Lichten, S. M., Bertiger, W. I., and Lindqwister, U. J., 1989. The effect of fiducial network strategy on high-accuracy orbit and baseline determination; in: W. L. Stein/M. Kumar (eds), *Proc. of the 5th Intl. Geodetic Symp. on Satellite Positioning*, Las Cruces, NM, USA. **I**: 516-525.
14. Papo, H. B., and Perlmuter, A., 1981. Datum definition by free net adjustment. *Bulletin Géodésique*, **55**: 218-226.
15. Perelmuter, A., 1979. Adjustment of free networks. *Bulletin Géodésique*, **53** (4): 291-295.
16. Schaffrin, B., 1984. On the use of prior information in free networks (in German); in: K. Rinner et al. (eds), *Ingenieur-Vermessung '84*, Dümmler: Bonn. **B6**:1-15.
17. Schaffrin, B., 1989. Fiducial versus fixed points in the GPS network approach; in: W. L. Stein/M. Kumar (eds.), *Proc. of the 5th. Intl. Geodetic Symp. on Satellite Positioning*, Las Cruces, NM, USA. **I**: 500-511.
18. Schaffrin, B., 1997. On suboptimal geodetic network fusion, Poster paper, *IAG General Meeting*, Rio de Janeiro, Brazil.
19. Schaffrin, B., 2002. Reproducing estimates via least squares: An optimal alternative to the Helmert transformation. In: E.W. Grafarend, F.W. Krumm, and V.S. Schwarz (eds.). Geodesy – The Challenge of the 3rd Millennium, Springer, Berlin/ Heidelberg/ New York: 387-392.
20. Schaffrin, B., and Cothren, J., 2003. Hierarchical data fusion with photogrammetric applications. *Brazilian Journal of Cartography*, **55** (2): 25-34.
21. Tan, W., 2002. Least squares adjustment through reparameterization. *Survey Review*, **36** (283): 362-366.
22. Welsch, W., 1979. Review of the adjustment of free networks. *Survey Review*, **24** (194): 167-180.