Deflection of the Vertical Components from GPS and Precise Leveling Measurements in Hong Kong

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Abstract: Deflection of the vertical components of a station located in Hong Kong are estimated from the ellipsoidal height differences inferred from global positioning system (GPS) measurements and orthometric height differences calculated from precise leveling using a simple formulation. The agreement of the deflection of the vertical components ξ and η obtained from the experiment (-7.3 ± 1.6, 5.3 ± 4.3 arc seconds, respectively) with the deflection of the vertical components calculated from the EGM96 model (-7.6, and 3.2 arc seconds, respectively) show that the approach can be used at a local scale using existing leveling networks to produce a detailed map of deflection of vertical components to be used in surveying applications and also for checking the reproducibility of potential alternative solutions.

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Introduction

The deflection of the vertical ϵ is the angle between the direction of the gravity vector **g** or plumb line at a point, and the ellipsoidal normal through the same point for a particular ellipsoid. It is conventionally decomposed into two perpendicular components: a north-south meridional component ξ and an east-west prime vertical component η.

The deflection of the vertical has applications in surveying activities such as reduction of directions, azimuths, zenith angles, and slope distances onto the ellipsoid.

Traditionally, deflection components are determined by astrogeodetic or gravimetric methods (Heiskanen and Moritz 1967). In the first method, the deflection of the vertical components are calculated using astronomical coordinates (Φ, Λ) and geodetic (ellipsoidal) coordinates (φ, λ). Gravimetric methods use Stokes' formula for calculation of the deflection of the vertical from gravity anomalies or harmonic expressions for the earth's gravity field.

Because global positioning system (GPS) and leveling measurements contain information about the ellipsoidal height and orthometric height, respectively, they can be used to determine deflection of the vertical components. Earlier studies by Soler et al. (1989), Vandenberg (1999), Fujii (1990), and Evans et al. (1989) show that calculation of deflection of verticals with the amalgamation of these new and traditional surveying technologies is possible. In this study we will investigate whether we can use GPS measurements and existing leveling networks to calculate deflection of vertical components in Hong Kong using some of the existing leveling networks. Because there is a multitude of GPS stations and already established leveling networks in Hong Kong, it is possible to determine deflection of the verticals at these stations and produce a detailed map of deflection of verticals to be used by the Hong Kong surveying community.

Differential Relationships and Methodology

The approach we will describe here is simple in concept and can be practical for determining deflection of vertical components in Hong Kong if the existing leveling networks and GPS measurements are deployed. The method can also be used to check the reproducibility of the deflection of vertical components to be determined using alternative methodologies.

Let us first look at the differential relationships between the geoid undulation and the deflection of the vertical (Heiskanen and Moritz 1967)

$$\epsilon = -\frac{dN}{ds} \quad (1)$$

The deflection of the vertical component ξ along the meridian and η along the prime vertical can be resolved along any geodetic azimuth α through

$$\epsilon = \xi \cos \alpha + \eta \sin \alpha \quad (2)$$

Combining both equations we obtain

$$-\frac{dN}{ds} = \xi \cos \alpha + \eta \sin \alpha \quad (3)$$

If we replace the differential elements that appear in these relationships with their discrete counterparts, we obtain

$$-\frac{\Delta N}{\Delta s} \approx \xi \cos \alpha + \eta \sin \alpha \quad (4)$$

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Now let us consider the geoid-ellipsoid separations (geoid undulations) at two closely spaced locations A and B on the surface of the earth expressed in terms of their orthometric H and ellipsoidal h heights

$$N_A = h_A - H_A \quad (5)$$

$$N_B = h_B - H_B \quad (6)$$

Subtracting Eq. (6) from (5), we obtain the undulation difference ΔN_{AB} between the two closely spaced points as

$$\Delta N_{AB} = N_A - N_B = h_A - H_B - (H_A - H_B) = \Delta h_{AB} - \Delta H_{AB} \quad (7)$$

Finally, substituting the Eq. (7) in Eq. (4), we get

$$-\frac{\Delta h_{AB} - \Delta H_{AB}}{\Delta s_{AB}} \approx \xi \cos \alpha_{AB} + \eta \sin \alpha_{AB} \quad (8)$$

The left-hand side of this expression can be obtained from precise leveling ΔH and GPS measurements Δh. Therefore, the deflection of the vertical components at a point on the earth can be estimated using several ancillary stations distributed around a point on the surface of the earth via a least squares solution.

Note that the preceding relationships hold if the separations between the master and the ancillary stations are very small (or the geoid undulations in the vicinity of the station vary smoothly—which is usually the case by their nature). Nevertheless, the larger the separation between two stations, the smaller the error of deflection of the vertical computed from the height difference observations, as shown subsequently.

Considering the left-hand side of Eq. (8)

$$\epsilon \approx -\frac{\Delta h - \Delta H}{\Delta s} \quad (9)$$

(subscripts dropped for clarity), using the variance propagation rule, and assuming that orthometric height differences and ellipsoidal height differences are not correlated, we get

$$\sigma_\epsilon^2 = \frac{1}{\Delta s^2}(\sigma_{\Delta H}^2 + \sigma_{\Delta h}^2) + \left(\frac{\Delta h - \Delta H}{\Delta s^2}\right)^2 \sigma_{\Delta s}^2 \quad (10)$$

The second term in parentheses in Eq. (10) is a fourth-order term and can be safely omitted. Hence

$$\sigma_\epsilon^2 = \frac{1}{\Delta s^2}(\sigma_{\Delta H}^2 + \sigma_{\Delta h}^2) \quad (11)$$

Let us quantify this expression. For $\Delta s = 1$ km and $\sigma_{\Delta H} \approx \sigma_{\Delta h} = \sigma$, we get

$$\sigma_\epsilon = \frac{\sqrt{12}}{10^6}\sigma \quad (12)$$

where σ is in mm. Hence, the error in determining the deflection of the vertical is linearly proportional to the errors of the GPS and precise leveling measurements. For closely spaced stations, both measurement techniques can be very accurate, because the systematic errors cancel out for GPS height differences and do not accumulate for the precise leveling observations. For instance, for $\sigma = 5$ mm, the standard deviation of the computed deflection of the vertical would be about $\sigma_\epsilon \approx 1.5$ arc second. The standard deviation of the estimated deflection of the vertical will reduce to $\sigma_\epsilon \approx 0.3$ arc second for $\sigma = 1$ mm.

In Hong Kong, the magnitudes of the deflection of the verticals are expected to be less than 15 arc seconds and the landscape is not too irregular, usually not exceeding a maximum of 600 m above the mean sea level over the territory. Hence, the geoid undulations vary smoothly over the region. With this information in mind, estimation of the deflection of the vertical from the observed orthometric height differences (precise leveling) and ellipsoidal heights differences (GPS) is feasible.

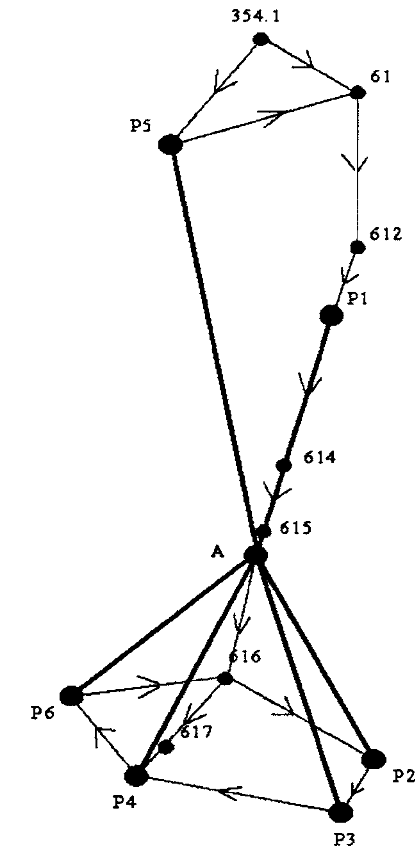

Fig. 1. Distribution of GPS and leveling stations (Tai Tong Road area in Yuen Long, Hong Kong)

Network Geometry and Determination of Height Differences

We selected a flat and open space location about 12 m above the Hong Kong Principal Datum that was suitable for both precise and GPS observations (Fig. 1). However, the ancillary stations around the test station were unevenly distributed due to difficulty of access for leveling measurements (remember that Hong Kong is one of the most densely populated areas in the world).

We used seven benchmarks from the Hong Kong Survey and Mapping Office as known points (354.1, 611–617) and ran six leveling routes in between these benchmarks to determine the orthometric height of stations P1–P6 and A as shown in Fig. 1. The leveling misclosures were within the adopted tolerance level of $4\sqrt{k}$ mm (k=leveling route distance in km). Sixteen height difference measurements resulted in six condition equations. The measurements were weighted inversely proportional to the distances between the benchmarks and assumed to be uncorrelated. After the adjustment, the residuals of the 16 height observations were less then 0.02 mm. We then calculated the orthometric height differences, ΔH (Table 1), and propagated their uncertainties to the orthometric height differences between stations P1–P6

Table 1. Orthometric and Ellipsoidal Height Differences between Stations P1 through P6 and Station *A*

From	To	Distance (m)	Azimuth (°)	ΔH (m)	Δh (m)
A	P1	906	14.3	−5.551	−5.555
A	P2	826	154.1	2.437	2.467
A	P3	1,008	164.5	4.187	4.215
A	P4	886	203.5	6.601	6.615
A	P5	1,529	350.5	−7.891	−7.973
A	P6	776	226.3	5.850	5.868

and station *A* to be used in the subsequent computation of the deflection of the verticals.

One Leica GPS System 300 and two Leica System 200s were used for the GPS measurements in static mode. In order to test the reproducibility of the GPS height measurements, a pair of Leica GPS System 200 and 300 receivers were deployed at a 300 m baseline for a 1 h measurement session. The GPS System 200 receiver was then replaced by another one for another 1 h session. Processed results show that the relative height measurements using two sequentially collocated receivers were consistent within 7 mm, which was not a very impressive outcome but was within acceptable limits of the simple analysis discussed earlier.

During the fieldwork, the GPS System 300 was deployed at reference station *A* and the two GPS System 200s were set at stations P1 and P5, which are approximately 1 km apart from the reference station *A*. They were then moved to stations P2 and P3, and finally P4 and P6. The processed ellipsoidal height differences are summarized in Table 1.

Estimation of Deflection of Vertical Components and Conclusion

We used Eq. (8) in the statistical model for computing the deflection of the vertical components from the height difference measurements between the two stations *A* and *B*

$$\frac{1}{\Delta s_{AB}}v_{\Delta h_{AB}} - \frac{1}{\Delta s_{AB}}v_{\Delta H_{AB}} + \cos\alpha_{AB}\xi + \sin\alpha_{AB}\eta + \frac{\Delta h_{AB} - \Delta H_{AB}}{\Delta s_{AB}} = 0 \quad (13)$$

Eq. (13) includes two deflections of the vertical components as unknown parameters, which can be estimated, and orthometric and ellipsoidal height difference observations, which can be adjusted using Helmert's "*conditioned observation equations with unknowns*" formulation (Iz 2002)

$$\boldsymbol{BV} + \boldsymbol{A}\hat{\boldsymbol{X}} + \boldsymbol{W} = 0 \quad (14)$$

where $\boldsymbol{B}=6\times 12$ matrix of coefficients for the residuals; $\boldsymbol{V}=12\times 1$ vector of residuals for the orthometric and ellipsoidal height differences between stations P1–P6 and station A; $\boldsymbol{A}=6\times 2$ design matrix; $\hat{\boldsymbol{X}}=2\times 1$ vector of unknown deflection of vertical components at station A; and $\boldsymbol{W}=6\times 1$ misclosure vector.

We assumed the GPS-derived ellipsoidal height differences to be $\sigma_{\Delta h}=0.02$ mm with an a priori variance of unit weight $\sigma_0^2=1$, and with no correlation between the observations. We used the full variance-covariance matrix derived after the adjustment of leveling measurements for the orthometric height differences. Furthermore, we assumed that the orthometric and ellipsoidal height observations were uncorrelated; i.e.

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_{\Delta h} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P}_{\Delta H} \end{pmatrix}_{12\times 12} \quad (15)$$

Please note that the solution has only four degrees of freedom due to the six conditioned equations and two unknown parameters.

The estimated parameters are 5.3 ± 4.3 and -7.3 ± 1.6 arc seconds for the east-west and north-south deflections of the vertical components, respectively. Their uncertainties are stated at the one sigma level. The a posteriori variance of unit weight for the solution is 0.9, indicating the use of an appropriate statistical model and weights for the solution. Although the residuals were at the submillimeter level for the orthometric height differences, some of the residuals reached the 2 cm level for the GPS-born ellipsoidal height differences (remember that the consistency check for the receivers was about 7 mm).

Because the degrees of freedom value for the solution was only four, any statistical inference based on null hypothesis testing would be meaningless. We therefore calculated the deflection of the vertical components at the same station using the EGM96 spherical harmonic model ("EGM96" 1998).

Table 2 shows that the north-south components from both measurement methods seem to agree with each other, whereas there is a significant difference in the east-west components from the two methods. Again, a statistical test is not meaningful because of the small degrees of freedom as well as the unknown accuracy of the EGM96 deflection of the vertical components in this region.

Nevertheless, part of the discrepancy in the east-west direction component can be explained by the distribution geometry of the ancillary stations around station *A*, as most of the stations were located in the north-south direction. We actually noted this potential problem during the reconnaissance stage by examining the a priori variance-covariance matrix of the estimates, but we were constrained in establishing a better geometry by the inaccessibility of more appropriate sites (pig farms in the region!) for leveling measurements.

Despite the apparent flaws in the experimental design, the agreement of the deflection of the vertical components from two completely independent approaches in both directions and, to a certain extent, in magnitudes are significant findings, demonstrating the ability of the method to resolve the deflection of the vertical components from GPS and precise leveling observations in local areas such as Hong Kong. Also, the use of existing first-order leveling networks supplemented by GPS observations makes the approach attractive among alternatives.

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References

"EGM96." (1998). *EGM Homepage*, ⟨http://cddisa.gsfc.nasa.gov/926/egm96/egm96.html/⟩.

Evans, A. G., Soler, T., Hothem, L. D., Stein, W. L., Hermann, B. R., and Carlson, A. E., Jr. (1989). "Vertical deflections and astronomic azimuth derived from GPS and leveling: Joint test description and results." *Proc., 5th Int. Geodetic Symp. on Satellite Positioning*, Physical Science Laboratory, New Mexico State Univ., Las Cruces, N.M., 945-955.

Fujii, Y. (1990). "Test observations and their results for determining geoidal height and deflection of vertical with GPS interferometry." *Japanese Symp. on Earth Rotation, Astrometry, and Geodesy*, 176-180 (in Japanese).

Heiskanen, W. A., and Moritz, H. (1967). *Physical geodesy*, Freeman, San Francisco.

Iz, H. B. (2002). "Analysis of Observation I Lecture Notes." *The Hong Kong Polytechnic University, Hong Kong*, ⟨http://elearning.lsgi.org/⟩.

Soler, T., Carlson, A. E. Jr., and Evans, A. G. (1989). "Determination of vertical deflections using the global positioning system and geodetic leveling." *Geophys. Res. Lett.*, 16(7), 695-698.

Vandenberg, D. J. (1999). "Combining GPS and terrestrial observations to determine deflections of the vertical." MS thesis, Purdue Univ., West Lafayette, Ind.