

Dependency of GPS positioning precision on station location

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Abstract We analyze the dependency of the GPS positioning precision on station location in an idealized positioning environment. Using a weighting function to account for an averaged number of satellites at different latitudes, we demonstrate that GPS positioning precision for ambiguity fixed solutions reduces when the latitude increases. We also quantify that the positioning precision in the north-south direction is worse than that in the east-west direction at all latitudes using the same approach.

Introduction

Santerre (1991) studied through simulation the impact of the observer's satellite sky distribution on the positioning with GPS which is dependent on the latitude of the observer's site. In this study we quantified this dependency in positioning using a weighting function to account for the availability of an average number of GPS satellites at different station locations.

We assume that the GPS satellite orbits are circular and the satellites move with uniform velocity. We first form the normal equations from the range observation equations from which the covariance matrix of the position errors are obtained. We then generalize the normal equations using a weighting function to account for the impact of the varying number of satellites available at

different latitudes. Using numerical quadrature we finally calculate and report the behavior of the point positioning errors for stations located at different latitudes, the behavior of GDOP, the correlations between station coordinates and clock offsets, and changes of the E-W and N-S components of positioning precision with station latitude.

Model of GPS observations and normal equations

The observation equation of GPS phase or pseudo-range can be written as (Hoffmann-Wellenhof et al. 1997)

$$v^j + \rho^j = \frac{X_0 - x^j}{\rho_0^j} \delta x + \frac{Y_0 - y^j}{\rho_0^j} \delta y + \frac{Z_0 - z^j}{\rho_0^j} \delta z - c\delta t + \lambda N^j + \rho_0^j \quad (1)$$

in which j is the number of the observed satellite, v^j is the residual, ρ^j is the range derived from code or phase observation for the j th satellite, δx , δy , δz are the corrections to a station's approximate coordinates, δt is the receiver and satellite clock correction, c is the nominal speed of light in a vacuum, ρ_0^j is the approximate distance between the station and the satellite computed using the station's approximate coordinates (X_0 Y_0 Z_0) and the satellite coordinates (x^j , y^j , z^j). N^j is the integer ambiguity and λ is the nominal wavelength of the carrier signal.

Let us define

$$\begin{aligned} \Delta x_0^j &= X_0 - x^j \\ \Delta y_0^j &= Y_0 - y^j \\ \Delta z_0^j &= Z_0 - z^j \\ \delta X &= (\delta X_0 \quad \delta Y_0 \quad \delta Z_0 \quad -c\delta t)^T \end{aligned} \quad (2)$$

Note that we will not consider ambiguity as an unknown parameter to only isolate the effect of the station position to the solution.

If we assume n satellites are observed, then the observation equations in a matrix/vector format are given by

$$V = A\delta X + L \quad (3)$$

where the coefficient (design) matrix A is

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$$A = \begin{bmatrix} \frac{\Delta x_0^1}{\rho_0^1} & \frac{\Delta y_0^1}{\rho_0^1} & \frac{\Delta z_0^1}{\rho_0^1} & -1 \\ \frac{\Delta x_0^2}{\rho_0^2} & \frac{\Delta y_0^2}{\rho_0^2} & \frac{\Delta z_0^2}{\rho_0^2} & -1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\Delta x_0^n}{\rho_0^n} & \frac{\Delta y_0^n}{\rho_0^n} & \frac{\Delta z_0^n}{\rho_0^n} & -1 \end{bmatrix} \quad (4)$$

In the above expression, X is the unknown parameter vector, δX is the vector of corrections to the nominal (approximate) values of the unknown parameters; L and V are the $n \times 1$ vector of observations and residuals, respectively. Using the above information, and assuming that the observations are equally weighted, we obtain the following symmetric coefficient matrix N of the normal equations:

$$N = A^T A = \begin{bmatrix} \sum_{j=1}^n \left(\frac{\Delta x_0^j}{\rho_0^j}\right)^2 & \sum_{j=1}^n \frac{\Delta x_0^j \cdot \Delta y_0^j}{(\rho_0^j)^2} & \sum_{j=1}^n \frac{\Delta x_0^j \cdot \Delta z_0^j}{(\rho_0^j)^2} & -\sum_{j=1}^n \frac{\Delta x_0^j}{\rho_0^j} \\ & \sum_{j=1}^n \left(\frac{\Delta y_0^j}{\rho_0^j}\right)^2 & \sum_{j=1}^n \frac{\Delta y_0^j \cdot \Delta z_0^j}{(\rho_0^j)^2} & -\sum_{j=1}^n \frac{\Delta y_0^j}{\rho_0^j} \\ & & \sum_{j=1}^n \left(\frac{\Delta z_0^j}{\rho_0^j}\right)^2 & -\sum_{j=1}^n \frac{\Delta z_0^j}{\rho_0^j} \\ & & & n \end{bmatrix} \quad (5)$$

At this point, we would like to generalize the normal equations for an arbitrary number of satellites and their locations. Because we do not know exactly how many satellites could be observed at a given time from a station location, we introduce $\bar{P}(\varphi)$, a weighting function which depends on the geocentric latitude φ of satellites to account for an average number of satellites. Modifying Eq. (5) we get

$$N = \begin{bmatrix} \iint_D \bar{P}(\varphi) \left(\frac{\Delta x_0^j}{\rho_0^j}\right)^2 d\varphi d\lambda & \iint_D \bar{P}(\varphi) \frac{\Delta x_0^j \cdot \Delta y_0^j}{(\rho_0^j)^2} d\varphi d\lambda & \iint_D \bar{P}(\varphi) \frac{\Delta x_0^j \cdot \Delta z_0^j}{(\rho_0^j)^2} d\varphi d\lambda & -\iint_D \bar{P}(\varphi) \frac{\Delta x_0^j}{\rho_0^j} d\varphi d\lambda \\ & \iint_D \bar{P}(\varphi) \left(\frac{\Delta y_0^j}{\rho_0^j}\right)^2 d\varphi d\lambda & \iint_D \bar{P}(\varphi) \frac{\Delta y_0^j \cdot \Delta z_0^j}{(\rho_0^j)^2} d\varphi d\lambda & -\iint_D \bar{P}(\varphi) \frac{\Delta y_0^j}{\rho_0^j} d\varphi d\lambda \\ & & \iint_D \bar{P}(\varphi) \left(\frac{\Delta z_0^j}{\rho_0^j}\right)^2 d\varphi d\lambda & -\iint_D \bar{P}(\varphi) \frac{\Delta z_0^j}{\rho_0^j} d\varphi d\lambda \\ & & & \iint_D \bar{P}(\varphi) d\varphi d\lambda \end{bmatrix} \quad (6)$$

where the integral area D is limited only above the horizon, i.e., above 15° elevation to reduce the impact of the severe atmospheric variations of the observations, and $d\varphi$ and $d\lambda$ are the sides of a differential integral area element in the sky bounded by geocentric latitudes and longitudes, respectively.

Satellite weighting function

We assume that the GPS satellites are equally distributed at six orbits with an inclination $i=55^\circ$. If GPS satellites are

distributed symmetrically in longitude direction, the satellite weighting function will only depend on the latitude. For perfectly circular satellite orbits, all satellites will run on a spherical surface with a radius $R=26,500$ km. Hence, a satellite's flight velocity is $2\pi R/T$, where T is the period of the satellite.

Because circular orbits on a sphere are geodesics, they satisfy the following Clairaut equation:

$$p \sin \alpha = \text{constant} = c \quad (7)$$

where p is the radius of a parallel circle, and α is the azimuth of geodetic line. For $i=55^\circ \rightarrow c=R \sin(90^\circ-i) = R \sin 35^\circ$. At a satellite position (φ, λ) where φ is geocentric latitude and λ is geocentric longitude, $p=R \cos \varphi$ and orbit azimuth α will satisfy the following expression:

$$R \cos \varphi \sin \alpha = R \sin 35^\circ \quad (8)$$

Hence, a satellite's angular velocity in the north-south direction V_φ and in the east-west direction V_λ can be written as

$$\begin{cases} V_\varphi = \frac{2\pi R \cos \alpha}{T} = k' \cos \alpha = k' \sqrt{1 - \left(\frac{\sin 35^\circ}{\cos \varphi}\right)^2} \\ V_\lambda = \frac{2\pi R \sin \alpha}{T R \cos \varphi} = k' \frac{\sin \alpha}{\cos \varphi} = k' \frac{\sin 35^\circ}{\cos^2 \varphi} \end{cases} \quad (9)$$

Since the faster a satellite moves, the smaller the probability that the satellite is available for observation, we define the following weighting function:

$$P(\varphi) = \frac{k''}{\sqrt{V_\varphi^2 + V_\lambda^2}} = k \frac{\cos^2 \varphi}{\sqrt{\cos^4 \varphi + \sin^2 35^\circ \cos^2 \varphi}} \quad (10)$$

where k' , k'' and k are constants.

To determine the constant k , consider the total number of 24 satellites for which

$$\iint_G P(\varphi) d\varphi d\lambda = 24 \quad (11)$$

where the integration is carried out over the sky G . If we divide the sky by $1^\circ \times 1^\circ$ blocks bounded by geocentric latitudes and longitudes, respectively, we obtain from Eq.(11) $k=0.00067656562112949$. Nevertheless, a GPS satellite can not have a latitude $|\varphi| > 55^\circ$. With this restriction we get a different result:

$$\bar{P}(\varphi) = \begin{cases} 0.00038183949259379^\circ \cos \varphi \sqrt{\cos^2 \varphi - \sin^2 35^\circ}, & |\varphi| < 55^\circ \\ 0, & |\varphi| \geq 55^\circ \end{cases} \quad (12)$$

Numerical quadrature for the variance/covariance matrix

Now we can compute Eq. (6) by dividing the whole sky into $1 \times 1^\circ$ blocks, and approximating each element of the coefficient matrix of the normal equations through the following equation:

$$\iint_D \bar{P}(\varphi) \left(\frac{\Delta x_0^j}{\rho_0^j} \right)^2 d\varphi d\lambda = \sum_{i=0}^{351} \sum_{k=0}^{89} \bar{P}(\varphi) \left(\frac{\Delta x_0^j}{\rho_0^j} \right)^2 \Delta\varphi \Delta\lambda \quad (13)$$

where D is the area confined to the elevation angle above 15° from the station's horizon, or $\bar{P}(\varphi)$ is set to zero when the satellite is below 15° mask angle.

For a station on an ellipsoidal earth with ellipsoidal latitude and longitude denoted by (B_0, L_0) and height $h_0=0$, the corresponding Cartesian coordinates are

$$\begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} = \begin{bmatrix} (R_N + h_0) \cos B_0 \cos L_0 \\ (R_N + h_0) \cos B_0 \sin L_0 \\ (R_N(1 - e^2) + h_0) \sin B_0 \end{bmatrix} \quad (14)$$

where $R_N = \frac{a}{\sqrt{1 - e^2 \sin^2 B_0}}$, a and e^2 are the prime vertical radius of curvature, semi major axis, and first eccentricity of the earth ellipsoid (WGS84 values used in numerical computations).

For a block (i, j) of a satellite in the sky, the satellite's Cartesian coordinates are given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} R \cos \varphi \cos \lambda \\ R \cos \varphi \sin \lambda \\ R \sin \varphi \end{bmatrix} \quad (15)$$

$$\frac{\partial(X_0 \ Y_0 \ Z_0)}{\partial(V_0 \ L_0 \ h_0)} = \begin{pmatrix} \frac{\partial R_N}{\partial B_0} \cos B_0 \cos L_0 - (R_N + h_0) \sin B_0 \cos L_0 & -(R_N + h_0) \cos B_0 \sin L_0 & \cos B_0 \cos L_0 \\ \frac{\partial R_N}{\partial B_0} \cos B_0 \sin L_0 - (R_N + h_0) \sin B_0 \sin L_0 & (R_N + h_0) \cos B_0 \cos L_0 & \cos B_0 \sin L_0 \\ \frac{\partial R_N}{\partial B_0} (1 - e^2) \sin B_0 + (R_N(1 - e^2) + h_0) \cos B_0 & 0 & \sin B_0 \end{pmatrix} \quad (20)$$

where $\varphi=i+0.5^\circ$, and $\lambda=j+0.5^\circ$. In order to calculate the satellite's elevation angle, we transform the satellite's Cartesian coordinates $(x' \ y' \ z')^T$ to the station's topocentric (up, north and east) coordinate system (U, N, E) using the following expression:

$$\begin{pmatrix} U \\ E \\ N \end{pmatrix} = R_2(-B_0)R_3(L_0) \begin{pmatrix} x' - X_0 \\ y' - Y_0 \\ z' - Z_0 \end{pmatrix} \quad (16)$$

from which the elevation angle θ is obtained as

$$\theta = \arcsin \frac{U}{\sqrt{U^2 + E^2 + N^2}} \quad (17)$$

Now the normal equations given by Eq. (6) can be calculated using the expressions through Eqs. (13), (14), (15), (16) and (17). When the elevation angle $\theta < 15^\circ$, the block (i, j) is out of the integral area D , or $\bar{P}(\varphi)$ in Eq. (13) is zero, otherwise $\bar{P}(\varphi)$ is calculated using Eq. (12).

For an a-priori variance of unit weight one, the inverse of numerically approximated Eq. (6) gives the variance/covariance matrix Q

$$Q = N^{-1} \quad (18)$$

in which, by definition, the geometric dilution of precision, GDOP, is the square root of the trace of the variance/covariance matrix Q . Note that the diagonal elements of the variance/covariance matrix refer to the variances, i.e., to the positioning precision in the Cartesian coordinates of a station. The corresponding latitude and longitude component precision can also be calculated by using the following expressions derived using variance propagation.

Let $(X_0, Y_0, Z_0)^T$ denote the Cartesian coordinates of a station given by Eq. (14). The uncertainties of the Cartesian coordinates can be transformed to the uncertainties of the geodetic coordinates $(B_0, L_0, h_0)^T$ as follows. Note that $h_0=0$ for station positions projected onto the ellipsoid.

Let $\delta X' = (\delta B_0, \delta L_0, \delta h_0, -c\delta t)^T$, then

$$\delta X = \begin{pmatrix} \frac{\partial(X_0 \ Y_0 \ Z_0)}{\partial(B_0 \ L_0 \ h_0)} & 0 \\ 0 & 1 \end{pmatrix} \delta X' = S \delta X' \quad (19)$$

with

in which

$$\frac{\partial R_N}{\partial B_0} = \frac{ae^2 \sin B_0 \cos B_0}{(1 - e^2(\sin B_0)^2)^{3/2}} \quad (21)$$

The covariance matrix Q_{BL} for $\delta X'$ can now be derived using the variance propagation rule as

$$Q_{BL} = S^{-1}Q(S^{-1})^T \quad (22)$$

The following expressions are obtained from the above expression to compute the ratio of the positioning precision in the E-W direction and in the N-S direction to the station's latitude which is independent of the scale as well as the units of the variance covariance matrix

$$\frac{R_N \cos B_0 \sqrt{Q_{LL}}}{R_M \sqrt{Q_{BB}}} \quad (23)$$

where R_M is the meridian radius of curvature.

Numerical results and conclusion

Figure 1 displays the variation of GDOP of a ground station at latitudes 0 to 89° with 1° intervals at 0° longitude.

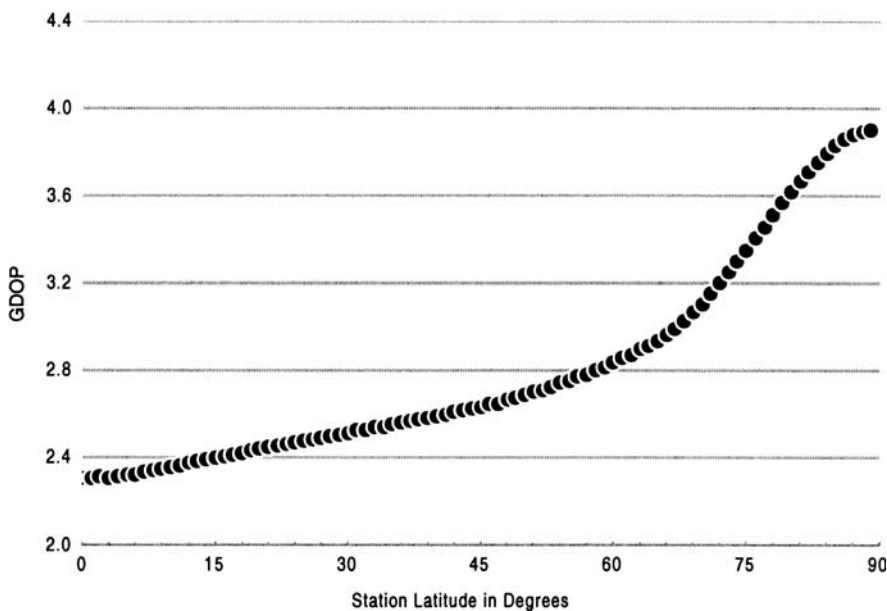


Fig. 1
The variation of GDOP as a function of station latitude

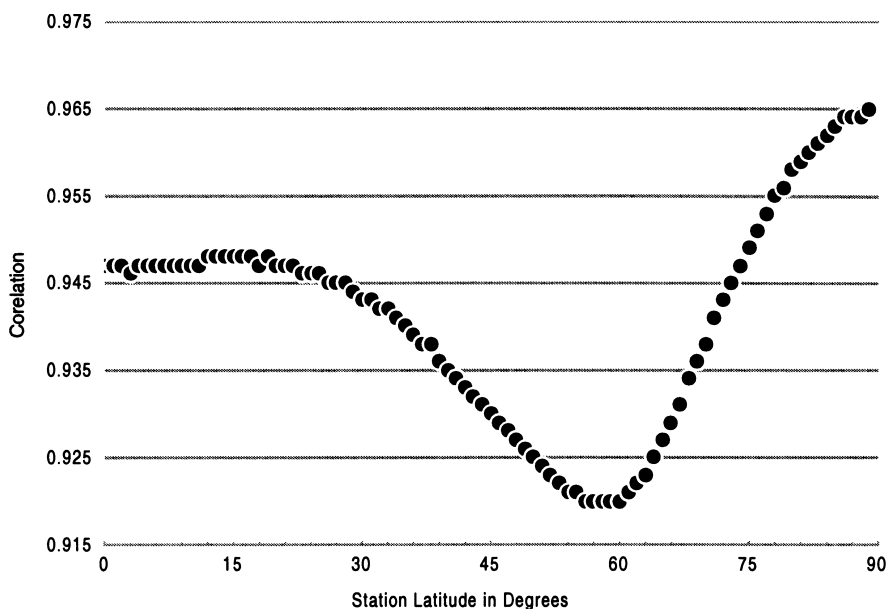


Fig. 2
Variation of the correlation between station height and receiver clock difference as a function of station latitude

This result shows that GDOP increases with increasing station latitude.

The variance/covariance matrix Q given by Eq. (18) is based on the Cartesian coordinates of a station. We express it in a geodetic coordinate system to investigate the correlation of geodetic height and receiver clock offset, and the change of positioning precision in the E-W and N-S components of a topocentric coordinate system as a function of station latitude.

Figure 2 depicts the variation of the correlation of station geodetic height h_0 , receiver/satellite clock difference $c\delta t$, as a function of station latitude B_0 . The significant correlation between GPS height and receiver clock difference is because both parameters share overlapping information. The results also reveal that the correlation is minimum for stations between latitudes 50–60°, but still remaining above 0.92.

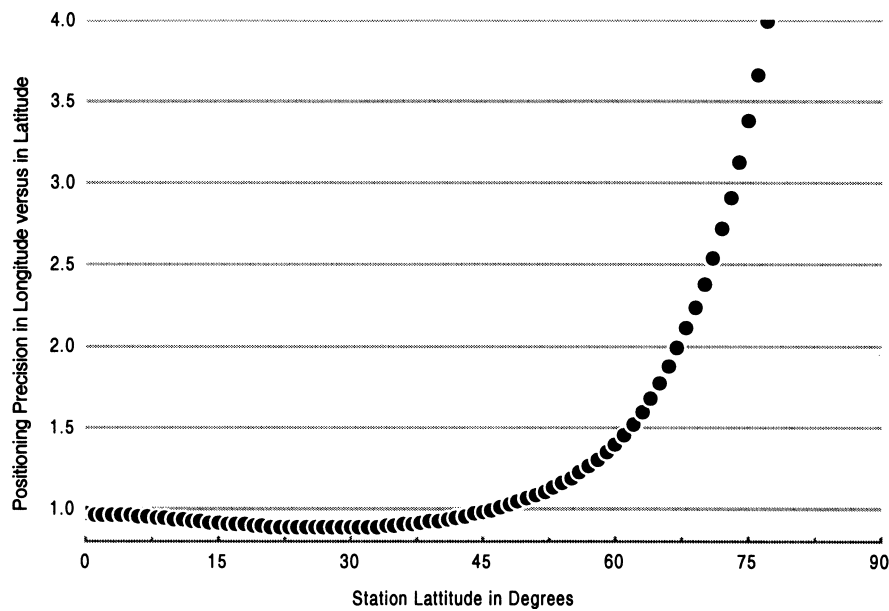


Fig. 3

The changes in the ratio of positioning precision of the longitude component to the positioning precision of the latitude component

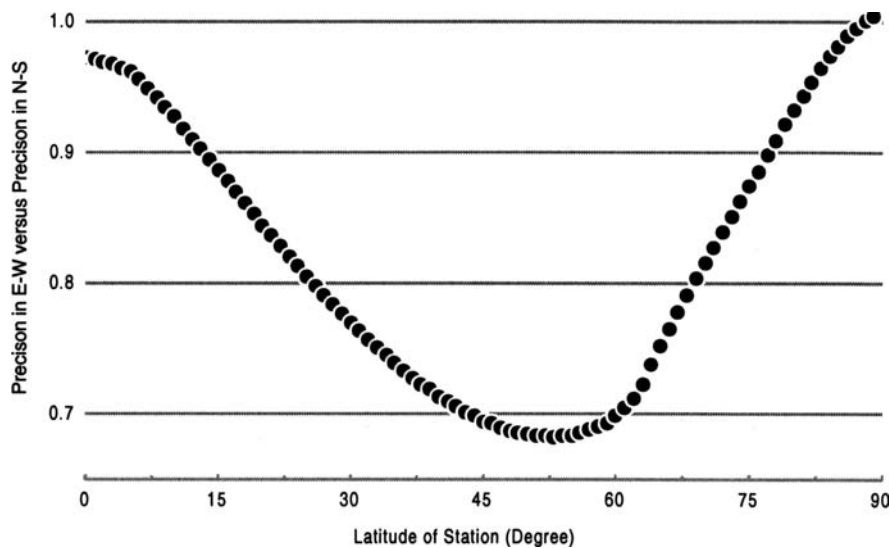


Fig. 4

The changes in the ratio of E-W positioning precision to the N-S positioning precision

Figure 3 shows the dependency of the ratio of the positioning precision in the longitude component to the latitude component precision, i.e., $\sqrt{Q_{LL}/Q_{BB}}$ to the station's latitude. The result reveals that the longitude component precision is slightly better than the error in the latitude component precision for stations with latitudes less than 45°. At higher station latitudes, the precision in longitude becomes larger and larger than the precision in the latitude. Figure 4 shows the dependency of the ratio of the positioning precision in the E-W direction and in the N-S direction to the station's latitude. The results indicate that the GPS positioning precision in the E-W direction is always smaller than the positioning precision component in the N-S direction.

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