Space-time Deformations of Global Tetrahedrons from USNO VLBI Baselines and their Contributions to the Terrestrial Reference Frame Realization

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Purpose

Establish reference frames attached to the VLBI stations forming fundamental tetrahedrons using only VLBI baseline observations and examine the evolution of their deformations.

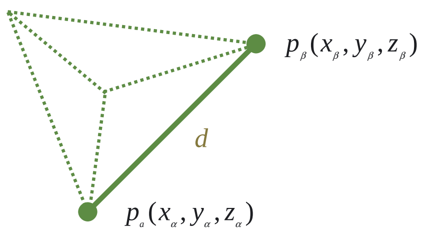
Presentation Outline

I. Mathematical model.
II. Statistical model.
III. Solution.
IV. Finite elements approach to analyze the solutions.
V. Numerical results and descriptive properties of the reference frames.

Fundamental Figures

One dimensional:	baseline	→	two stations.
Two dimensional:	triangle	→	three stations.
Three dimensional:	tetrahedron	→	four stations.

Mathematical Model



$$d_{\alpha\beta}^2 = (x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2 + (z_\alpha - z_\beta)^2$$

Statistical Model

$$y = F(\mathbf{x}) + u$$

$$u \sim (0, \Sigma_u)$$

- *y* is the *nx1* vector of observed baseline lengths.
- *u is* the *nx1* vector of disturbances with zero mean and a positive definite variance/covariance matrix.
- *F*(***x***) is the nonlinear function of distance.
- ***x*** is the *mx1* vector of unknown parameters (station coordinates = no. of stations x 3).

Linearized Model

$$\begin{aligned} d_{\alpha\beta_{observed}} = d_{0\alpha\beta} &+ \left.\frac{\partial d_{\alpha\beta}}{\partial x_\alpha}\right|_0 (x_\alpha - x_{0\alpha}) + \left.\frac{\partial d_{\alpha\beta}}{\partial y_\alpha}\right|_0 (y_\alpha - y_{0\alpha}) + \left.\frac{\partial d_{\alpha\beta}}{\partial z_\alpha}\right|_0 (z_\alpha - z_{0\alpha}) + \\ &\left.\frac{\partial d_{\alpha\beta}}{\partial x_\beta}\right|_0 (x_\beta - x_{0\beta}) + \left.\frac{\partial d_{\alpha\beta}}{\partial y_\beta}\right|_0 (y_\beta - y_{0\beta}) + \left.\frac{\partial d_{\alpha\beta}}{\partial z_\beta}\right|_0 (z_\beta - z_{0\beta}) + u \end{aligned}$$

For a tetrahedron:

6 baselines → *y* is a 6x1 vector of baseline lengths.
4 stations x 3 → 12 unknowns (corrections to the nominal values).

Hence there are 6 observation equations with 18 unknowns (12 corrections and 6 unknown observation errors).

Minimum Norm Least Squares Solution (MINOLESS)

Constructing Normal equations through the Least Squares formalism reduces the number of unknowns to 12.

$$\Delta\mathbf{y} = \mathbf{A}\Delta\mathbf{x} + \mathbf{u}$$

where,

$\Delta\mathbf{y}_{6x1} := d_{\text{observed}} - d_{\text{nominal}}$

$\mathbf{A}_{6x12}$ coefficient (design) matrix - from partials.

$\Delta\mathbf{x}_{12x1} := \mathbf{x} - \mathbf{x}_{\text{nominal}}$

Normal Equations

$$(\mathbf{A}^T\boldsymbol{\Sigma}_u^{-1}\mathbf{A})\Delta x = \mathbf{A}^T\boldsymbol{\Sigma}_u^{-1}\Delta\mathbf{y}$$

The rank of the normal matrix = 6, the dimension of the normal matrix = 12, hence there are multitude of solutions that satisfy the above system. We choose the solution which has the minimum norm, as well as minimum variance.

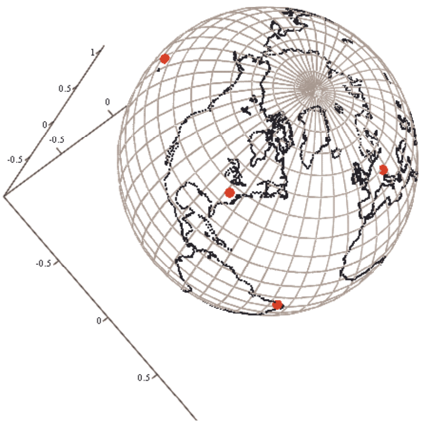
$$\Delta x = (\mathbf{A}^T\boldsymbol{\Sigma}_u^{-1}\mathbf{A})^{+}\mathbf{A}^T\boldsymbol{\Sigma}_u^{-1}\Delta\mathbf{y}$$

The above solution is iterated until there are no changes in the baseline lengths. Note that in this particular case, the input and output baseline lengths remain the same because there are no redundancies in the figure for an adjustment.

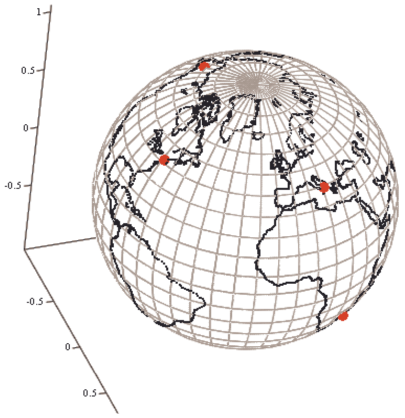
Solution Properties

- Minimum variance solution, $trD(\hat{x}) = \min$.
- Minimum norm solution, $|\Delta x| = \min$ (*Auffelderung*, Wolf 1973).
- The corrections to the approximate coordinates of the center of mass of the tetrahedron are zero => the origin is fixed.
- The rotation of the tetrahedron to be estimated with respect to the tetrahedron defined by the approximate (nominal) coordinates is zero (Koch 1987). Also known as no net rotation solution.

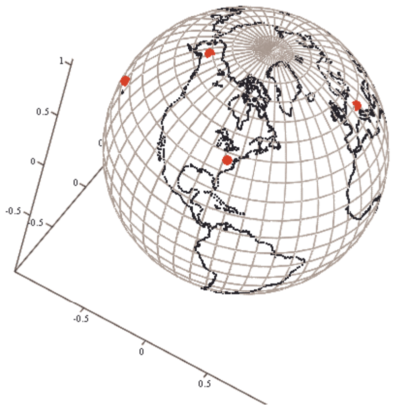
Network: FKNW (Fortleza, Kokee, NRAO20, Wettzell).



Network: GHMW (Gilcreek, Hartrao, Matera, Westford).



Network: GKNW (Gilcreek, Kokee, NRAO20, Wettzell).



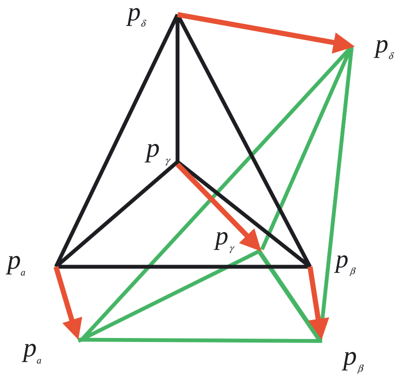
Time series Evolution of the Station Coordinates

- Baselines are derived from the Navy 1999-3 solution. Only simultaneously observed baselines are used for each tetrahedron (not corrected for plate motions).
- Approximate coordinates of the stations are defined by the above solution coordinates at the beginning epoch of each tetrahedron baseline observations.
- Solutions include the variance/covariance matrices of the baseline lengths.

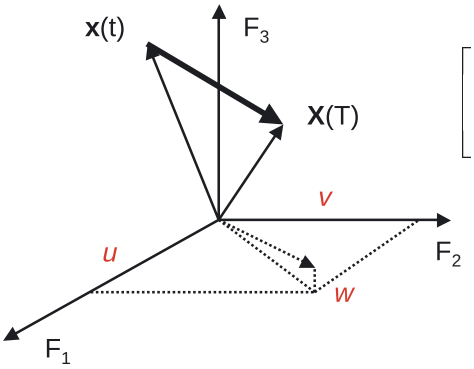
Analysis of VLBI Baselines Using Strain Tensor Elements

- Infinitesimal deformations.
- Strain tensor elements.
- Cumulative deformation of the fundamental tetrahedron.

Deformation of the Fundamental Tetrahedron



Displacement Vector



$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} := \begin{bmatrix} X(T) - x(t) \\ Y(T) - y(t) \\ Z(T) - z(t) \end{bmatrix}$$

Approximating with a Taylor point (chosen to be the mass or geometric center of the tetrahedron)

$$\begin{aligned} u(x,y,z) = u(x_0,y_0,z_0) &+ u_x(x_0,y_0,z_0)(x-x_0) \\ &+ u_y(x_0,y_0,z_0)(y-y_0) \\ &+ u_z(x_0,y_0,z_0)(z-z_0) \end{aligned}$$

$$\begin{aligned} v(x,y,z) = v(x_0,y_0,z_0) &+ v_x(x_0,y_0,z_0)(x-x_0) \\ &+ v_y(x_0,y_0,z_0)(y-y_0) \\ &+ v_z(x_0,y_0,z_0)(z-z_0) \end{aligned}$$

$$\begin{aligned} w(x,y,z) = w(x_0,y_0,z_0) &+ w_x(x_0,y_0,z_0)(x-x_0) \\ &+ w_y(x_0,y_0,z_0)(y-y_0) \\ &+ w_z(x_0,y_0,z_0)(z-z_0) \end{aligned}$$

Where the mass center of the tetrahedron is given by

$$x_0 = \frac{1}{4}(x_\alpha + x_\beta + x_\delta + x_\gamma)$$

$$y_0 = \frac{1}{4}(y_\alpha + y_\beta + y_\delta + y_\gamma)$$

$$z_0 = \frac{1}{4}(z_\alpha + z_\beta + z_\delta + z_\gamma)$$

Solution

$$\begin{bmatrix} u_\alpha & v_\alpha & w_\alpha \\ u_\beta & v_\beta & w_\beta \\ u_\gamma & v_\gamma & w_\gamma \\ u_\delta & v_\delta & w_\delta \end{bmatrix} = \begin{bmatrix} 1 & x_\alpha - x_0 & y_\alpha - y_0 & z_\alpha - z_0 \\ 1 & x_\beta - x_0 & y_\beta - y_0 & z_\beta - z_0 \\ 1 & x_\gamma - x_0 & y_\gamma - y_0 & z_\gamma - z_0 \\ 1 & x_\delta - x_0 & y_\delta - y_0 & z_\delta - z_0 \end{bmatrix} \begin{bmatrix} u_{\alpha x} & v_{\alpha y} & w_{\alpha z} \\ u_{\beta x} & v_{\beta y} & w_{\beta z} \\ u_{\gamma x} & v_{\gamma y} & w_{\gamma z} \\ u_{\delta x} & v_{\delta y} & w_{\delta z} \end{bmatrix}$$

$U = MA$

If V denotes the volume of the tetrahedron then,

$|M| = 6\,V$

Description of the Infinitesimal Deformations of the Fundamental Tetrahedron

Normal strain

$$e_{11} = u_x = \frac{\partial u}{\partial x} \quad e_{22} = v_y = \frac{\partial v}{\partial y} \quad e_{33} = w_z = \frac{\partial w}{\partial z}$$

Shear

$$e_{12} = \frac{1}{2}(u_y + v_x) \quad e_{13} = \frac{1}{2}(u_z + w_x) \quad e_{23} = \frac{1}{2}(v_z + w_y)$$

Rotation

$$\omega_{12} = \frac{1}{2}(u_y - v_x) \quad \omega_{13} = \frac{1}{2}(u_z - w_x) \quad \omega_{23} = \frac{1}{2}(v_z - w_y)$$

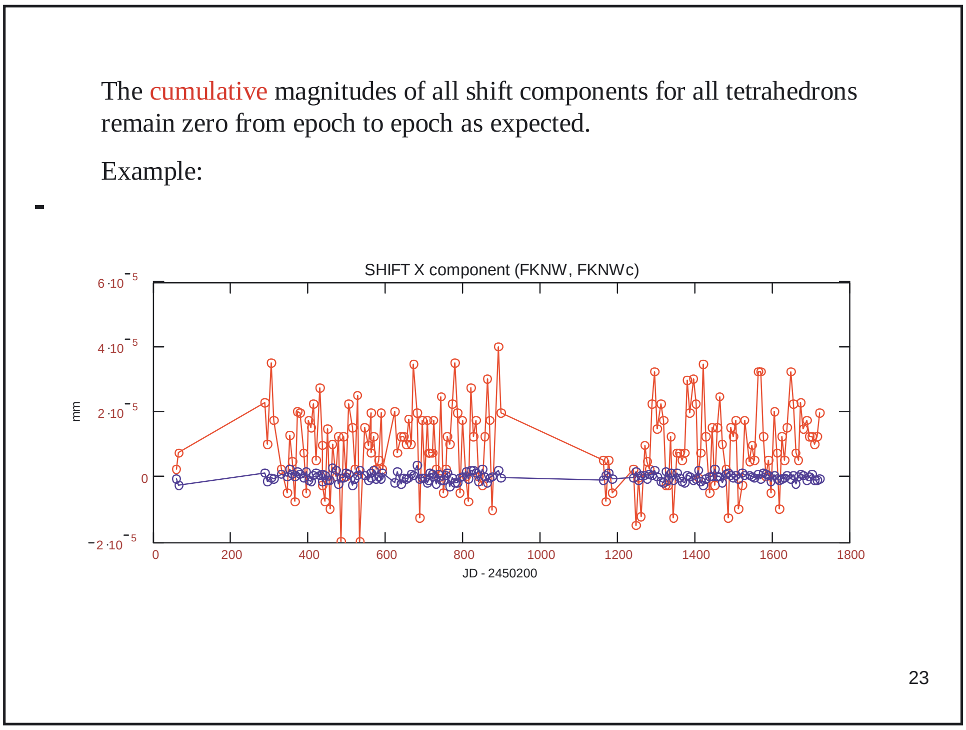
Translation

$$u_0 = u(x_0, y_0, z_0) \quad v_0 = v(x_0, y_0, z_0) \quad w_0 = w(x_0, y_0, z_0)$$

Strain Tensor Elements

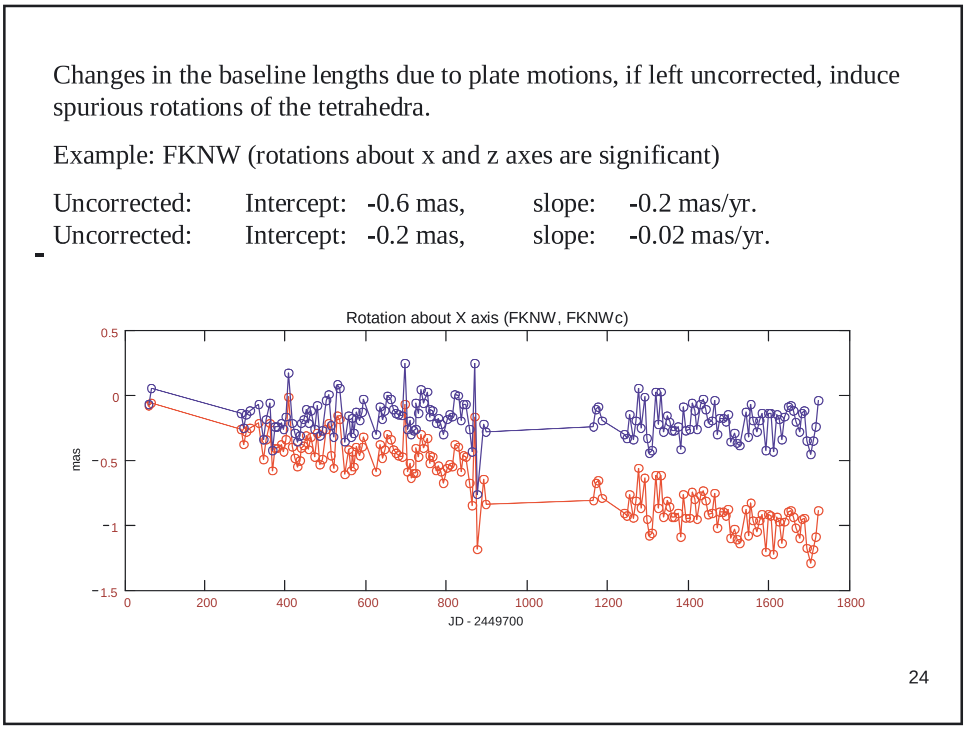
$$e_{ij} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} \quad \omega_{ij} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

Numerical Results



The cumulative magnitudes of all shift components for all tetrahedrons remain zero from epoch to epoch as expected.

Example:



Changes in the baseline lengths due to plate motions, if left uncorrected, induce spurious rotations of the tetrahedra.

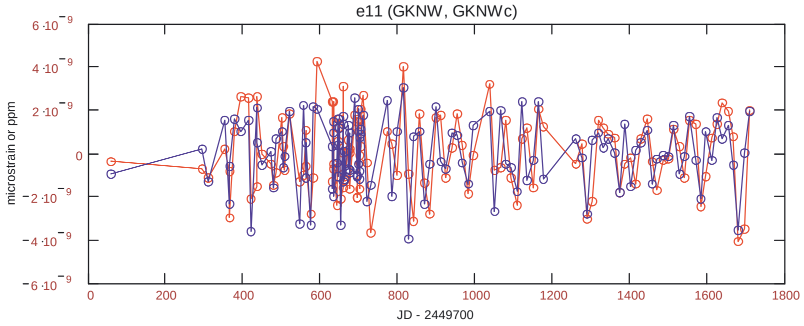
Example: FKNW (rotations about x and z axes are significant)

Uncorrected:	Intercept:	-0.6 mas,	slope:	-0.2 mas/yr.
Uncorrected:	Intercept:	-0.2 mas,	slope:	-0.02 mas/yr.

The deformations in terms of normal strain in the x direction for all tetrahedra are not cumulative and the impact of plate motion is negligibly small. The variability for all tetrahedra is about 1.0E-9 microstrain.

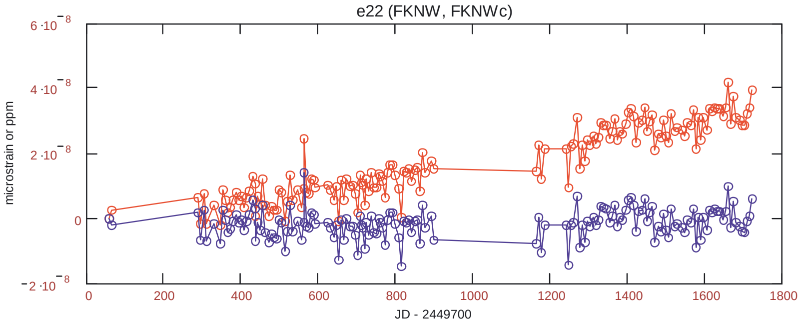
Example:

Uncorrected (in microstrain): mean: -1.9E-11, stdev: 1.69E-9
Corrected (in microstrain): mean: -4.1E-12, stdev: 1.54E-9



The variability of the normal strain component in the y direction is about 1.0E-9 µstrain for all tetrahedra. Changes in the baselines lengths due to plate motions induce significant deformations only in the FKNW.

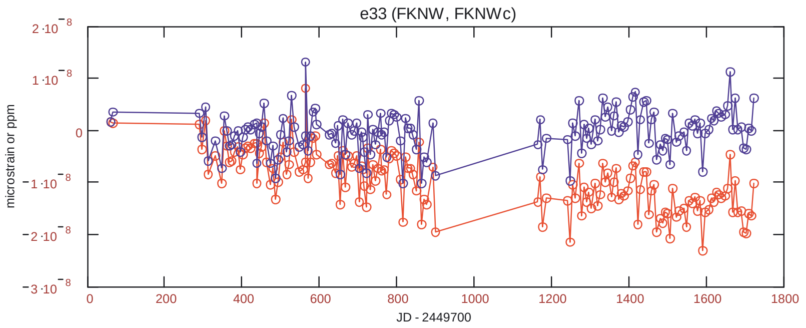
Uncorrected: Int.: 1.7E-8 µstrain, slope: 7.90E-9 µstrain/yr
Corrected: Int.: -1.6E-9 µstrain, slope: 6.71E-10 µstrain/yr.



The variability of the normal strain component in the z direction is about 1.0E-9 µstrain for all tetrahedrons. Changes in the baseline lengths due to the plate motions induce significant deformations only in the FKNW.

Example:

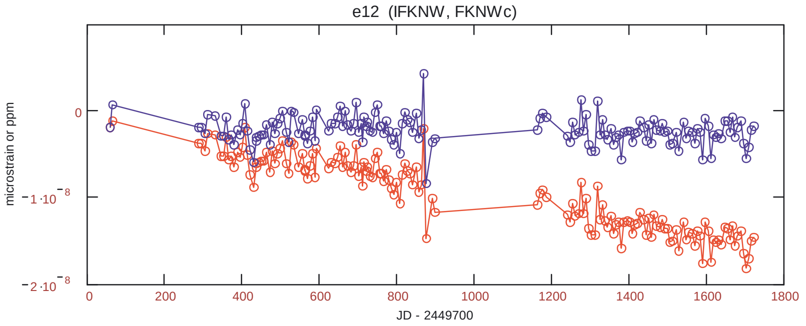
Uncorrected:	Int.: -9.5E-9 µstrain,	slope: -3.00E-9 µstrain/yr.
Corrected:	Int.: -3.0E-10 µstrain,	slope: 5.70E-10µstrain/yr.



Again, the variability of the shear component in the xy direction is about 1.0E-9 µstrain for all tetrahedrons. Changes in the baseline lengths due to the plate motions induce significant deformations only in the FKNW.

Example:

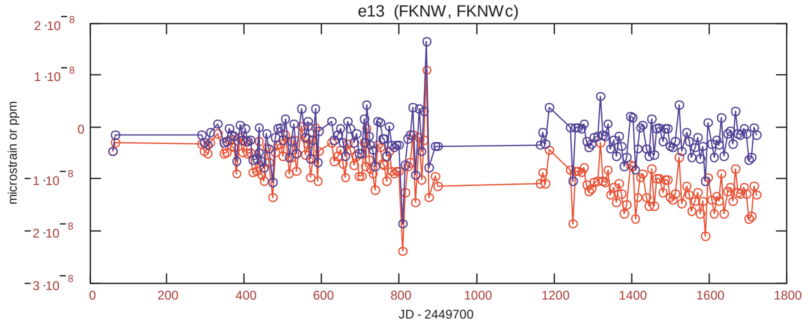
Uncorrected:	Int.: -9.5E-9 µstrain,	slope: -3.03E-9 µstrain/yr.
Corrected:	Int.: -2.2E-9µstrain,	slope: -2.33E-10µstrain/yr.



Again, the variability of the shear component in the xz direction is about 10E-9 µstrain for all tetrahedrons. Changes in the baseline lengths due to the plate motions induce significant deformations only in the FKNW.

Example:

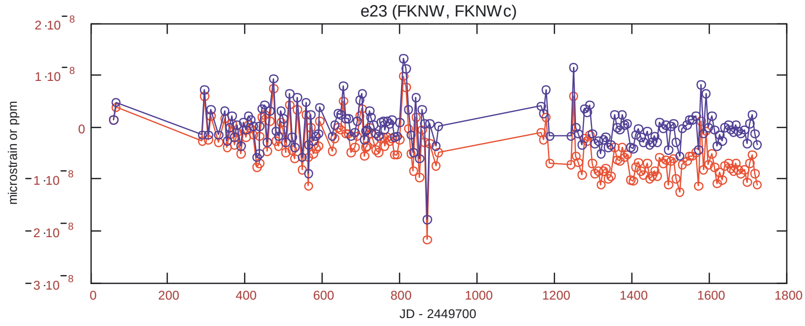
Uncorrected:	Int.: -8.8E-9µstrain,	slope: -2.47E-9µstrain/yr.
Corrected:	Int.: -2.3E-9µstrain,	slope: 2.99E-11µstrain/yr.



The variability of the shear component in the yz direction is about 10E-9 µstrain for all tetrahedrons. Changes in the baseline lengths due to the plate motions induce significant deformations only on FKNW.

Example:

Uncorrected:	Int.: -4.31E-9 µstrain,	slope: -2.10E-9 µstrain/yr.
Corrected:	Int.: 5.67E-11 µstrain,	slope: -4.11E-10 µstrain/yr.



Concluding Remarks

- Station motions due to plate motions (PM) deform all tetrahedra. FKNW tetrahedron deforms significantly compared to the other two. Dispersion for all configurations (corrected for PM) is 1.0E-9 µstrain for the stain tensor elements, and a few mas for the rotations.
- There are no translations on all tetrahedra because of the solution constraint.
- Tensor elements contain more information about the deformations than the rigid body motions only.
- PM corrected tetrahedral deformations are NOT cumulative but contain hidden periodicities at shorter time intervals due to tidal loading at the stations, residual atmospheric effects on VLBI baselines…, (Iz and Chen 1999).
- Further coordinate stability is possible from polyhedral solutions. They can be used as reference solutions to investigate the systematic effects present in the coordinate dependent solutions.